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ANALYSIS OF FRAME-REINFORCED CYLINDRICAL SHELLS

PART II - DISCONTINUITIES OF CIRCUMFERENTIAL - BENDING STIFFNESS IN THE AXIAL DIRECTION

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SUMMARY

The stress distribution in, and adjacent to, an externally-loaded frame in a cylindrical shell is extended to include the effects of discontinuities of circumferential-bending stiffness in the axial direction. These effects may be caused by nearby heavy frames, planes of symmetry and antisymmetry, and free ends. Such problems can be solved with the aid of the "transmission" matrix for a finite length of shell. A complete derivation for the elements of this matrix is given, which defines the force-displacement relationships at the ends of a finite length of shell. In addition to indicating exact solutions, this report derives an approximate technique and applies it to a number of practical problems.

NOTATION

A_{ik}	matrix elements defined in equation (100)
a_n	$(n^2 - 1)(L_r/L_c)^2/3$
B_{ik}	matrix elements defined in equation (119)
$[C_n(x)]$	matrix defined in equation (41)
$[D]$	transmission matrix for a frame
E	Young's modulus (lbs/in^2)
f_{in}	abbreviations defined in equation (66)
G	shear modulus (lbs/in^2)
H_{ik}	matrix elements defined in equation (110)

[1]Originally prepared as LMSD 49733, Lockheed Missiles and Space Division, Sunnyvale, California, and reproduced in original form by NASA, by agreement with Lockheed Aircraft Corporation, to increase availability.

I_ℓ	moment of inertia of frame at $x = \ell$ (in^4)
i	I/ℓ_o (in^3)
J_1	$R_2 + R_4$
J_2	$R_2 - R_4$
j	$\sqrt{-1}$
K	abbreviation defined in equation (58)
K_n	$n\sqrt{n^2 - 1}\,(1 + 2a_n) \Big/ 2\sqrt{3}\sqrt{1 + a_n}$
L_c	characteristic length (see glossary) $= r\left[t'r^2/i\right] \Big/ \sqrt{6}$ (in.)
L_r	characteristic length (see glossary) $= r\sqrt{Et'/Gt}/2$ (in.)
ℓ	distance from externally loaded frame
ℓ_o	frame spacing (in.)
M_o	externally applied concentrated moment (in. - lbs)
M_{ik}	element of transmission matrix defined in equation (18)
N_1	$R_1 + R_3$
N_2	$R_1 - R_3$
N_c	parameter defined in equation (38) of reference 1
n	index of harmonic depedance in the ϕ direction
P_o	externally applied concentrated radial load (lbs)
P^*	abbreviation defined in equation (116)
P_{in}	roots of the characteristic equation
Q	abbreviation for $Et'r\,Z_{12}/n(W + Z_{11})$
Q_i	matrix elements defined in equation (30)
q	shear flow in skin (lbs/in)
R	abbreviation for $Et'r^2 W Z_{11} / n^2(W + Z_{11})$
R_i	matrix elements (see equation [29])

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r	radius of the skin line (in.)
S_i	matrix elements defined in equation (31)
$[S_n(x)]$	matrix defined in equation (41)
T_o	externally applied concentrated tangential load (lbs)
T_{ik}	elements of transmission matrix defined in equation (19)
t	skin panel thickness (in.)
t'	effective skin panel thickness for axial loads (in.)
U_i	matrix elements defined in equation (32)
u	axial displacement (in.)
V_{ik}	matrix elements defined in equation (120)
v	tangential displacement
W	abbreviation defined in equation (114)
w	radial displacement (in.)
X_{ik}	matrix elements defined in equation (122)
X_{ik}^*	matrix elements defined in equation (127)
X_{ik}'	matrix elements defined in equation (129)
X_{ik}''	matrix elements defined in equation (131)
x	axial coordinate of the shell (in.)
Z	abbreviation defined in equation (96)
Z_{ik}	elements of transmission matrix defined by equations (27) and (102)
Z^*	abbreviation defined in equation (116)
α_n	real part of the complex roots of the characteristic equation
α_{in}	real roots of the characteristic equation
β_n	imaginary part of the complex roots of the characteristic equation
γ	"beef-up" parameter $= I_0/2iL_c$
Δq	shear flow applied to the frame (lbs/in)
Γ	see equation (60)
$Re\Gamma$	real part of Γ
$Im\Gamma$	imaginary part of Γ

σ_{in}	abbreviations defined in equation (34)
$[\lambda]$	matrix defined in equation (41)
$[\Omega]$	matrix defined in equation (41)

GLOSSARY OF TERMINOLOGY

The terms "Input Impedance", "Transmission Matrix" and "Characteristic Length" are used in this report and are defined as follows:

Input Impedance is the relationship between the tangential displacement and shear flow harmonic coefficients of the shell at the section of the loaded frame.

Transmission Matrix. The forces and displacements at one end of a finite length of unloaded shell can be written in terms of their values at the other end. The square matrix defining these relationships is the Transmission Matrix.

Characteristic Length. In this report there are two Characteristic Lengths, defined as follows. L_c is the distance required for the exponential envelope of the lowest order self equilibrating stress system to decay to $1/e$ of its value at $x = 0$, provided that the skin panels are rigid in shear. L_T is the distance required for the envelope of the lowest order self equilibrating stress system to decay to $1/e$ of its value at $x = 0$, provided that the frames are rigid in bending.

INTRODUCTION

In many practical shell problems, the shell is uniform in the circumferential direction but varies discontinuously in the axial direction. These discontinuities may be caused by free ends, rigid bulkheads, planes of symmetry or antisymmetry, and frames whose bending stiffness is much larger than the typical frames. In Part I of this report (ref. 1), a stress-and-deflection analysis is derived for frame-supported shells on the assumption that non-externally-loaded frames could be "smeared out" in the axial direction, thus producing an infinitely long shell of uniform circumferential-bending stiffness. This assumption is shown to introduce a considerable simplification in obtaining the desired results for shells whose bending stiffness does not vary greatly from frame to frame, except for the externally-loaded frame. A simple correction is derived to account for finite frame spacing. The analysis enables tables of coefficients for the computation of loads and deflections to be presented as a function of one parameter, γ, in reference 2.

Clearly, the model on which these results are based is inadequate where there are large variations in frame-bending stiffness near the loaded frame. A shell with marked discontinuities in frame-bending stiffness can be looked upon as being a number of finite lengths of shell (in which secondary frames are "smeared out," as before) attached at their ends to the concentrated frames that cause the discontinuities in stiffness. Free ends, and planes of symmetry and antisymmetry cause similar discontinuities in circumferential-bending stiffness.

In this report, the analysis of a finite length of shell is undertaken and the input-output relations are derived. These relations are defined by the elements of a transmission matrix. The transmission matrix for a frame is also given. Once these matrices are known, the problem at hand can be solved in a simple and rational manner by applying the physical boundary conditions to the relevant matrix elements.

In addition to indicating the exact solutions, an approximate method is proposed for correcting the basic parameter, γ , of references 1 and 2, to account for the type of shell problem indicated. This enables the tables of reference 2 to be used directly for a wide variety of problems by using a value of γ modified as indicated in this report.

The derived methods frequently rely heavily on results obtained in reference 1, and it is assumed that the reader is familiar with that report.

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GENERAL FORM OF THE SOLUTION

If load is applied to a single frame, the remainder of the shell enters the solution by means of the relationship between tangential displacement at the loaded frame and the net shear flow applied to the loaded frame by the shell. Let the net shear flow acting on the loaded frame be:

$$\Delta q = q(o^+) - q(o^-) \tag{1}$$

The symmetric net shear flow harmonic coefficient is

$$\overline{\Delta q}_n = \bar{q}_n(o^+) - \bar{q}_n(o^-) \tag{2}$$

By analogy with equation (49) of reference 1, the relationship between tangential displacement and net shear flow will be:

$$\frac{\bar{v}_n(o)}{\overline{\Delta q}_n(o)} = \frac{r^4}{Ei(n^3 - n)^2} \cdot \frac{K_n}{2L_c} \cdot f(n) \tag{3}$$

It can be easily proved that f(n) is a function of n such that

$$\lim_{n \to \infty} f(n) = \text{constant} = \left\{ \begin{array}{l} 1 \text{ for non-end frame} \\ 2 \text{ for end frame } L_r/L_c \neq 0 \\ 4 \text{ for end frame } L_r/L_c = 0 \end{array} \right\}$$

provided that adjacent bays have "smeared out" frames.

Using the definition for the characteristic length, L_c, we have $r^4/i = 36\ L_c^4/t'r^2$. Substituting this expression into equation (3), a more useful equation for f(n) is derived, i.e.:

$$f(n) = \frac{\bar{v}_n(o)}{\overline{\Delta q}_n} \cdot \frac{Et'r^2(n^3 - n)^2}{36 L_c^3} \cdot \frac{2}{K_n} \tag{4}$$

The proof lies in the fact that nearby discontinuities in the shell have less and less effect on the higher self-equilibrating stress systems, and for very large values of n, even a nearby discontinuity appears to be a large distance away. Hence, if the loaded frame is not at the end of the shell, f(n) must approach one, because equation (3) is just equation (49) of reference 1, with $f(n) = 1$ and $\overline{\Delta q}_n = 2\ \bar{q}_n(o)$. If the loaded frame is at the end of the shell, f(n) must approach two, if $L_r/L_c \neq 0$, because equation (3) is just equation (B.7) of reference 1 with $f(n) = 4\ \alpha_n L_c/K_n$, which approaches two for $n \rightarrow \infty$. For $L_r/L_c = 0$, $\frac{4\ \alpha_n L_c}{K_n}$ approaches four as $n \rightarrow \infty$.

The general form of the solutions for concentrated loads can consequently be written from equations (66) and (67) of reference 1 as:

$$\overline{\Delta q}_n = \frac{\frac{nP_o}{\pi r}}{1 + \gamma K_n f(n)} \tag{5}$$

$$\overline{\overline{\Delta q}}_n = \frac{-1}{\pi r} \left\{ \frac{T_o + \frac{(1 - n^2)}{r} M_o}{1 + \gamma K_n f(n)} \right\} \tag{6}$$

Expressions for forces and displacements in the loaded frame and loads per inch in the shell are given in terms of $\bar{q}_n(o)$ and $\bar{\bar{q}}_n(o)$ in reference 1. The ratios $\bar{q}_n(o)/\overline{\Delta q}_n$ and $\bar{\bar{q}}_n(o)/\overline{\overline{\Delta q}}_n$ are found for a number of typical situations later on in this report. Hence, complete solutions can be generated for each problem. This method, while yielding an exact solution, must be solved for each particular problem, since there are too many parameters involved to cover the range of practical problems with a reasonable number of charts or tables.

In equations (5) and (6) it is noted that f(n) always occurs multiplied by γ. This raises the possibility that if f(n) does not vary greatly with n, it is reasonable to use f(2) instead of f(n) for all values of n in equations (5) and (6). The great utility of such an approximation, if valid, is that the loads and displacements in the loaded frame and loads per inch in the shell can be obtained from the tables of reference 2 by using $f(2) \cdot \gamma$ in place of γ.

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As an introduction to assessing the validity of this approximation, it is instructive to evaluate the function $f(n) = 4\ \alpha_n\ L_c / K_n$ which applies to the problem of a reinforced loaded frame at the free end of a semi-infinite shell. Utilizing the defining formulas for α_n and K_n :

$$f(n) = 4 \left[\frac{1 + \frac{n^2-1}{3}\left(\frac{L_r}{L_c}\right)^2}{1 + \frac{2}{3}(n^2-1)\left(\frac{L_r}{L_c}\right)^2} \right] \tag{7}$$

The following table lists $f(2)$ and the ratio $f(n)/f(2)$ for various values of n and L_r/L_c .

$\frac{Lr}{Lc}$	$f(2)$	$\frac{f(3)}{f(2)}$	$\frac{f(4)}{f(2)}$	$\frac{f(5)}{f(2)}$	$\frac{f(\infty)}{f(2)}$
0	4.00	1.00	1.00	1.00	.500
.3	3.70	.908	.826	.763	.541
.5	3.33	.857	.772	.689	.600
.707	3.00	.820	.778	.718	.667
1.00	2.67	.868	.816	.779	.750
∞	2.00	1.000	1.000	1.000	1.000

Observe that for $n \leq 5$, the maximum change in $f(n)/f(2)$ is of the order of 30 percent. If $f(2)$ were substituted for $f(n)$ in equations (5) and (6), the net effect on the solution would be very small, since:

(1) The stress system for $n = 2$ tends to dominate the solution.

(2) Differences between $f(n)$ and $f(2)$ are of the order of 30 percent for $n \leq 5$.

(3) Changes in γ , and hence, changes in $f(3)$, $f(4)$, etc., of the order of 30 percent, produce small changes in the solution.

For the case of the non-end frame, it has been shown that $\lim_{n \to \infty} f(n) = 1.0$. It is reasonable to expect that for $n = 3, 4, 5,$ etc., $f(n)$ should lie between $f(2)$ and 1. Hence, it is possible to establish a criterion for the substitution of $f(2)$ for $f(n)$ in equation (3). This criterion is tentatively taken to be that for a "non-end" frame, the substitution will be acceptable if $1/4 < f(2) < 4$, and for an end frame, the substitution will be acceptable if $1/2 < f(2) < 8$. Otherwise it is necessary to compute the individual corrections for each significant higher-order stress system.

This simplified approximation permits the treatment of the effect of nearby discontinuities by means of the basic tables (ref. 2), plus simple graphs giving $f(2)$ as a function of distance to the nearest discontinuity. For the derivations of these graphs it is necessary to develop the concept of a transmission matrix, relating the input-output loads and deflections for a finite length of shell. However, a note of caution should be sounded at this stage. The tables of reference 2 are derived for shells that are symmetrical about the loaded frame. Therefore, the loads per inch in the shell computed by the approximate method indicated may be in error for shells that violate that symmetry condition. The further from the loaded frame that the cause of unsymmetry occurs, the less significant does the error become.

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TRANSMISSION MATRICES

Boundary Conditions and Introduction of Transmission Coefficients

In addition to relationships between quantities existing at opposite ends of a finite length of shell, relationships between quantities on the opposite sides of a point of discontinuity must be written. The most general discontinuity to be considered is a flexible frame shown in figure 1. In writing relationships between forces and displacements, harmonic coefficients may be substituted for the forces and displacements (ref. 1). The equations for the above figure are:

$$\bar{p}_n(x_1^-) = \bar{p}_n(x_1^+) \tag{8}$$

$$\overline{\Delta q}_n(x_1) = \bar{q}_n(x_1^+) - \bar{q}_n(x_1^-) \tag{9}$$

$$\bar{u}_n(x_1^-) = \bar{u}_n(x_1^+) \tag{10}$$

$$\bar{v}_n(x_1) = \bar{v}_n(x_1^-) = \bar{v}_n(x_1^+) \tag{11}$$

$$\bar{v}_n(x_1) = -\frac{r^4}{E I_1 (n^3 - n)^2} \overline{\Delta q}_n(x_1) \tag{12}$$

Equation (12) follows from equation (62) of reference 1, with $P_o = 0$.

Similar equations exist for the antisymmetric coefficients. For simplicity, only the symmetric case is considered here, but the results also apply for antisymmetric coefficients.

Important limiting cases are obtained by setting I_1 equal to zero or infinity.

In some cases it is convenient to use, in the part of the shell to the left of the loaded frame, a coordinate x increasing to the left (the coordinate system for the shell

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to the right of the frame is rotated 180 degrees about a vertical axis). The only difficulties arise in connection with boundary conditions at the loaded frame, which are illustrated in figure 2.

The boundary conditions are:

$$p(0,\phi) = p(0^*, -\phi) \tag{13}$$

$$\Delta q(0,\phi) = q(0,\phi) - q(0^*, -\phi) \tag{14}$$

$$u(0,\phi) = -u(0^*, -\phi) \tag{15}$$

$$v(0,\phi) = -v(0^*, -\phi) \tag{16}$$

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For symmetrical loading:

$$\left.\begin{aligned} p(0^*, -\phi) &= p(0^*, \phi) \\ q(0^*, -\phi) &= -q(0^*, \phi) \\ u(0^*, -\phi) &= u(0^*, \phi) \\ v(0^*, -\phi) &= -v(0^*, \phi) \end{aligned}\right\} \tag{17}$$

With these substitutions, trigonometric terms can be eliminated, leaving the following relationships between harmonic coefficients:

$$\bar{p}_n(o) = \bar{p}_n(o^*)$$

$$\overline{\Delta q}_n = \bar{q}_n(o) + \bar{q}_n(o^*)$$

$$\bar{u}_n(o) = -\bar{u}_n(o^*)$$

$$\bar{v}_n(o) = \bar{v}_n(o^*)$$

The relationships between force and displacement harmonic coefficients at the ends $(x = 0$ and $x = \ell)$ of a finite segment of shell are conveniently given by a 4×4 matrix of transmission coefficients having the following form:

$$\left|\begin{matrix} \bar{p}_n(\ell) \\ \bar{q}_n(\ell) \\ \bar{u}_n(\ell) \\ \bar{v}_n(\ell) \end{matrix}\right| = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \left|\begin{matrix} \bar{p}_n(o) \\ \bar{q}_n(o) \\ \bar{u}_n(o) \\ \bar{v}_n(o) \end{matrix}\right| \tag{18}$$

This form of the relationship is convenient for theoretical discussion and as a starting form for analysis. Later, the following form will be introduced:

$$\left|\begin{matrix} \bar{q}_n(\ell) \\ \frac{Et'r}{n}\,\bar{u}_n(\ell) \\ \frac{r}{n}\,\bar{p}_n(\ell) \\ \frac{Et'r^2}{n^2}\,\bar{v}_n(\ell) \end{matrix}\right| = \left[\begin{array}{cc|cc} T_1 & T_2 & T_5 & T_6 \\ T_3 & T_4 & T_7 & T_8 \\ \hline T_9 & T_{10} & T_{13} & T_{14} \\ T_{11} & T_{12} & T_{15} & T_{16} \end{array}\right] \left|\begin{matrix} \bar{q}_n(o) \\ \frac{Et'r}{n}\,\bar{u}_n(o) \\ \frac{r}{n}\,\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\,\bar{v}_n(o) \end{matrix}\right| \tag{19}$$

The boundary conditions of an unloaded frame can also be written in the form of a transmission matrix. Substitute equation (12) into equation (9) and write equations (8) through (11) as a single matrix equation:

$$\left|\begin{matrix} \bar{p}_n(x_1^+) \\ \bar{q}_n(x_1^+) \\ \bar{u}_n(x_1^+) \\ \bar{v}_n(x_1^+) \end{matrix}\right| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{-EI_1(n^3-n)^2}{r^4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left|\begin{matrix} \bar{p}_n(x_1^-) \\ \bar{q}_n(x_1^-) \\ \bar{u}_n(x_1^-) \\ \bar{v}_n(x_1^-) \end{matrix}\right| \tag{20}$$

Relationships between quantities existing at any two locations in the shell can be obtained by multiplying transmission matrices for the intervening frames and shell lengths (figure 3).

Let Φ (x) stand for the column of four harmonic coefficients at x . Let transmission matrices for the bays be represented by $[M]_1$ and $[M]_2$.

Let transmission matrices for the frames be represented by $[D]_o$, $[D]_1$, and $[D]_2$.

Then

$$\left|\Phi(x_2^+)\right| = [D]_2 [M]_2 [D]_1 [M]_1 [D]_o \left|\Phi(o)\right| \tag{21}$$

In this part of the report, transmission coefficients are derived for a finite length of shell. The use of such coefficients for the solution of problems is treated in the third section.

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GENERAL PROPERTIES OF TRANSMISSION MATRICES

It is evident, from Maxwell's reciprocation law, that not more than ten of the coefficients given in equation (18) are independent. If the equations of dependence can be discovered, they can be used to eliminate calculation of some of the elements (if they are simple equations) and to check the operations involved in obtaining overall transmission matrices, such as equation (21). The properties of transmission matrices are discussed briefly in reference 3, pages 76 to 87. The inverse of a transmission matrix can be obtained without numerical computations as shown in reference 3. In order to make use of this property, equation (18) must be slightly rewritten:

$$\begin{vmatrix} -\bar{p}_n(\ell) \\ \bar{q}_n(\ell) \\ \bar{u}_n(\ell) \\ \bar{v}_n(\ell) \end{vmatrix} = \begin{bmatrix} M_{11} & -M_{12} & -M_{13} & -M_{14} \\ -M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{31} & M_{32} & M_{33} & M_{34} \\ -M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{vmatrix} -\bar{p}_n(o) \\ \bar{q}_n(o) \\ \bar{u}_n(o) \\ \bar{v}_n(o) \end{vmatrix} \tag{22}$$

Then, according to the result given in reference (3):

$$\begin{vmatrix} -\bar{p}_n(o) \\ \bar{q}_n(o) \\ \bar{u}_n(o) \\ \bar{v}_n(o) \end{vmatrix} = \begin{bmatrix} M_{33} & M_{43} & M_{13} & -M_{23} \\ M_{34} & M_{44} & M_{14} & -M_{24} \\ M_{31} & M_{41} & M_{11} & -M_{21} \\ -M_{32} & -M_{42} & -M_{12} & M_{22} \end{bmatrix} \begin{vmatrix} -\bar{p}_n(\ell) \\ \bar{q}_n(\ell) \\ \bar{u}_n(\ell) \\ \bar{v}_n(\ell) \end{vmatrix} \tag{23}$$

Applying this result to equation (19), there results:

$$
\begin{vmatrix} \bar{q}_n(o) \\ \frac{Et'r}{n}\bar{u}_n(o) \\ \frac{r}{n}\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\bar{v}_n(o) \end{vmatrix}
=
\begin{bmatrix} T_{16} & T_{14} & -T_8 & -T_6 \\ T_{15} & T_{13} & -T_7 & -T_5 \\ -T_{12} & -T_{10} & T_4 & T_2 \\ -T_{11} & -T_9 & T_3 & T_1 \end{bmatrix}
\begin{vmatrix} \bar{q}_n(\ell) \\ \frac{Et'r}{n}\bar{u}_n(\ell) \\ \frac{r}{n}\bar{p}_n(\ell) \\ \frac{Et'r^2}{n^2}\bar{v}_n(\ell) \end{vmatrix}
\tag{24}
$$

If the object to which the transmission matrix refers is symmetrical with respect to its ends, the inverse of the transmission matrix may also be obtained by principles of symmetry. For example, consider a finite length of shell, loaded as shown in view (a) of figure 4.

The loading and displacements are symmetrical with respect to the plane $\phi = 0$ so that the loading system shown corresponds to symmetrical harmonic coefficients. If the shell is rotated through 180 degrees about a vertical axis, it will appear as in view (b) of figure 4.

If the directions of $q(\ell)$, $u(\ell)$, $q(o)$ and $u(o)$ are reversed in view (b), the loading will be the same as that in view (a), with (o) and (ℓ) interchanged. Since the appearance of the structure is indistinguishable in the two cases, due to its symmetry, equation (19) may be used with the following interchanges:

$$\bar{q}_n(\ell) \longleftrightarrow -\bar{q}_n(o)$$

$$\bar{u}_n(\ell) \longleftrightarrow -\bar{u}_n(o)$$

$$\bar{p}_n(\ell) \longleftrightarrow \bar{p}_n(o)$$

$$\bar{v}_n(\ell) \longleftrightarrow \bar{v}_n(o)$$

The result is:

$$
\begin{vmatrix} \bar{q}_n(o) \\ \frac{Et'r}{n}\bar{u}_n(o) \\ \frac{r}{n}\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\bar{v}_n(o) \end{vmatrix}
=
\begin{bmatrix} T_1 & T_2 & -T_5 & -T_6 \\ T_3 & T_4 & -T_7 & -T_8 \\ -T_9 & -T_{10} & T_{13} & T_{14} \\ -T_{11} & -T_{12} & T_{15} & T_{16} \end{bmatrix}
\begin{vmatrix} \bar{q}_n(\ell) \\ \frac{Et'r}{n}\bar{u}_n(\ell) \\ \frac{r}{n}\bar{p}_n(\ell) \\ \frac{Et'r^2}{n^2}\bar{v}_n(\ell) \end{vmatrix}
\tag{25}
$$

By comparison of equation (25) with equation (4), the six following identities result:

$$\left.\begin{aligned}
T_{16} &= T_1 \\
T_{15} &= T_3 \\
T_{14} &= T_2 \\
T_{13} &= T_4 \\
T_{12} &= T_9 \\
T_8 &= T_5
\end{aligned}\right\} \tag{26}$$

This list does not exhaust the dependent relationships between transmission coefficients. The assumption of symmetry introduced three additional constraints in addition to the six introduced by reciprocity. A search for these additional relationships is not made. For any particular type of object, additional accidental relationships between coefficients may exist. The six relationships given above are used in later sections of this report.

Input Impedance for a Semi-Infinite Shell

For a semi-infinite shell segment, relationships must be obtained between the four quantities at the near end of the shell. These relationships have already been obtained in Appendix B of reference 1. They are, rewritten in the style of equation (19):

$$\begin{vmatrix} \dfrac{Et'r^2}{n^2}\,\bar{v}_n(o) \\ \\ \dfrac{Et'r}{n}\,\bar{u}_n(o) \end{vmatrix} = \begin{bmatrix} \dfrac{72\,L_c^4\alpha_n}{n^4(n^2-1)^2} & \dfrac{6\,L_c^2}{n^2(n^2-1)} \\ \\ \dfrac{-6\,L_c^2}{n^2(n^2-1)} & \dfrac{-12\,L_c^2\alpha_n}{n^2(n^2-1)} \end{bmatrix} \begin{vmatrix} \bar{q}_n(o) \\ \\ \dfrac{r}{n}\,\bar{p}_n(o) \end{vmatrix} \tag{27}$$

Equation (27) is sometimes more convenient in the form:

$$\begin{vmatrix} \bar{v}_n(o) \\ \bar{u}_n(o) \end{vmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{vmatrix} \bar{q}_n(o) \\ \bar{p}_n(o) \end{vmatrix}$$

where

$$Z_{11} = \frac{72L_c^4\,\alpha_n}{n^2(n^2-1)Et'r} \qquad Z_{12} = -Z_{21} = \frac{6L_c^2}{n(n^2-1)Et'r} \tag{27a}$$

and

$$Z_{22} = -\frac{12\,L_c^2\alpha_n}{n^2(n^2-1)Et'}$$

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Transmission Coefficients for a Finite Length of Shell Without Stiffening Frames

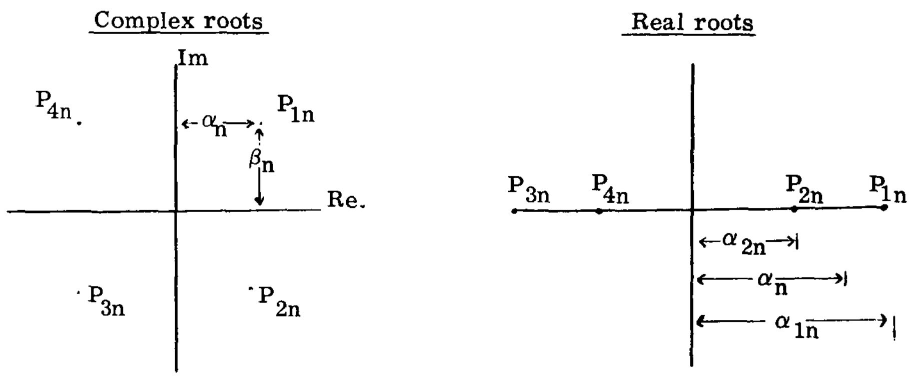
These transmission coefficients have already been obtained in Appendix A of reference 1. Rewrite equations (A.6), (A.7), (A.8) and (A.9) of reference 1 in the style of equation (19):

$$\begin{Bmatrix} \bar{q}_n(\ell) \\ \frac{Et'r}{n}\bar{u}_n(\ell) \\ \frac{r}{n}\bar{p}_n(\ell) \\ \frac{Et'r^2}{n^2}\bar{v}_n(\ell) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2}\ell^2 & 1 & \ell & 0 \\ \ell & 0 & 1 & 0 \\ \left(\frac{\ell^3}{6} - \frac{r^2 \ell Et'}{n^2 Gt}\right) & \ell & \frac{1}{2}\ell & 1 \end{bmatrix} \begin{Bmatrix} \bar{q}_n(o) \\ \frac{Et'r}{n}\bar{u}_n(o) \\ \frac{r}{n}\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\bar{v}_n(o) \end{Bmatrix} \tag{28}$$

Note that the identities of equation (26) are all satisfied. Note also that the coefficients all have the dimension of various powers of length.

Transmission Coefficients of a Finite Length of Shell with Stiffening Frames

The transmission coefficients are obtained by solving the differential equations for a shell with "smeared-out" frames and evaluating the constants of integration in terms of $\bar{p}_n(o)$, $\bar{q}_n(o)$, $\bar{u}_n(o)$, and $\bar{v}_n(o)$ for the symmetrical case. The constants of integration are exactly the same for the antisymmetric case. In reference 1 this is done for a shell extending to infinity and, in so doing, terms with increasing exponential factors are eliminated. In treating a finite length of shell, all four roots (P_{1n}, P_{2n}, P_{3n}, and P_{4n}) of the characteristic equation must be retained. The following diagrams show the location of these roots.



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$n < N_c$	$n > N_c$
$P_{1n} = \alpha_n + j\beta_n$	$P_{1n} = \alpha_{1n}$
$P_{2n} = \alpha_n - j\beta_n$	$P_{2n} = \alpha_{2n}$
$P_{3n} = -\alpha_n - j\beta_n = -P_{1n}$	$P_{3n} = -\alpha_{1n} = -P_{1n}$
$P_{4n} = -\alpha_n + j\beta_n = -P_{2n}$	$P_{4n} = -\alpha_{2n} = -P_{2n}$

Each of the four quantities, $\bar{p}_n(x)$, $\bar{q}_n(x)$, $\bar{u}_n(x)$, and $\bar{v}_n(x)$, can be expressed as exponential functions of the roots of the characteristic equations multiplied by undetermined coefficients:

$$\begin{vmatrix} \bar{p}_n(x) \\ \bar{q}_n(x) \\ \bar{u}_n(x) \\ \bar{v}_n(x) \end{vmatrix} = \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ R_1 & R_2 & R_3 & R_4 \\ S_1 & S_2 & S_3 & S_4 \\ U_1 & U_2 & U_3 & U_4 \end{bmatrix} \begin{vmatrix} e^{P_{1n}x} \\ e^{P_{2n}x} \\ e^{P_{3n}x} \\ e^{P_{4n}x} \end{vmatrix} \tag{29}$$

The number of undetermined coefficients can be reduced to four by means of equations (17) through (19) of reference 1.

The four general solutions satisfy these equations independently. Let P_{in} stand for any one of the four roots, and Q_i, R_i, S_i, and U_i stand for the corresponding undetermined coefficients. Then, by substitution of a column of equation (29) into equations (17) through (19) of reference 1:

$$Q_i = \frac{n\,R_i}{r\,P_{in}} \tag{30}$$

$$S_i = \frac{1}{Et'P_{in}} \cdot Q_i = \frac{nR_i}{Et'r\,P_{in}^2} \tag{31}$$

$$U_i = \frac{1}{P_{in}}\left(\frac{nS_i}{r} - \frac{1}{Gt}\,R_i\right) = \frac{n^2 R_i}{Et'r^2 P_{in}}\left[\frac{1}{P_{in}^2} - \frac{4L_r^2}{n^2}\right] \tag{32}$$

These results may now be substituted into equation (29). Some simplification results if common factors are cleared to the left-hand side, and if rows are interchanged:

$$\begin{vmatrix} \bar{q}_n(x) \\ \frac{Et'r}{n}\,\bar{u}_n(x) \\ \frac{r}{n}\,\bar{p}_n(x) \\ \frac{Et'r^2}{n^2}\,\bar{v}_n(x) \end{vmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{P_{1n}^2} & \frac{1}{P_{2n}^2} & \frac{1}{P_{3n}^2} & \frac{1}{P_{4n}^2} \\ \frac{1}{P_{1n}} & \frac{1}{P_{2n}} & \frac{1}{P_{3n}} & \frac{1}{P_{4n}} \\ \frac{\sigma_{1n}}{P_{1n}} & \frac{\sigma_{2n}}{P_{2n}} & \frac{\sigma_{3n}}{P_{3n}} & \frac{\sigma_{4n}}{P_{4n}} \end{bmatrix} \begin{vmatrix} R_1 e^{P_{1n}x} \\ R_2 e^{P_{2}x} \\ R_3 e^{P_{3n}x} \\ R_4 e^{P_{4n}x} \end{vmatrix} \tag{33}$$

where

$$\left.\begin{aligned} \sigma_{1n} &= \frac{1}{P_{1n}^2} - \frac{4\,L_r^2}{n^2}\,, \\ \sigma_{2n} &= \frac{1}{P_{2n}^2} - \frac{4\,L_r^2}{n^2}\,, \;\ldots \end{aligned}\right\} \tag{34}$$

It is convenient to substitute hyperbolic functions for the exponential functions. This is aided by the fact that, whether the roots are real or complex, $P_{3n} = -P_{1n}$ and $P_{4n} = -P_{2n}$. Hence,

$$R_1 e^{P_{1n}x} + R_3 e^{P_{3n}x} = N_1 \cosh P_{1n}x + N_2 \sinh P_{1n}x \tag{35}$$

where

$$N_1 = R_1 + R_3 \quad \text{and} \quad N_2 = R_1 - R_3 \tag{36}$$

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Similarly,

$$R_2 e^{P_{2n}x} + R_4 e^{P_{4n}x} = J_1 \cosh P_{2n}x + J_2 \sinh P_{2n}x \tag{37}$$

where

$$J_1 = R_2 + R_4 \quad \text{and} \quad J_2 = R_2 - R_4 \tag{38}$$

Other terms in equation (33) can be similarly combined. The various coefficients are:

Quantity	Coefficient of cosh $P_{1n}x$	Coefficient of sinh $P_{1n}x$
$\bar{u}_n(x)$	$\frac{R_1}{P_{1n}^2} + \frac{R_3}{P_{3n}^2} = \frac{N_1}{P_{1n}^2}$	$\frac{R_1}{P_{1n}^2} - \frac{R_3}{P_{3n}^2} = \frac{N_2}{P_{1n}^2}$
$\bar{p}_n(x)$	$\frac{R_1}{P_{1n}} + \frac{R_3}{P_{3n}} = \frac{N_2}{P_{1n}}$	$\frac{R_1}{P_{1n}} - \frac{R_3}{P_{3n}} = \frac{N_1}{P_{1n}}$
$\bar{v}_n(x)$	$\frac{R_1\sigma_{1n}}{P_{1n}} + \frac{R_3\sigma_{3n}}{P_{3n}} = \frac{\sigma_{1n}N_2}{P_{1n}}$	$\frac{R_1\sigma_{1n}}{P_{1n}} - \frac{R_3\sigma_{3n}}{P_{3n}} = \frac{\sigma_{1n}N_2}{P_{1n}}$

Combine these results into a new matrix equation:

$$\begin{Vmatrix} \bar{q}_n(x) \\ \frac{Et'r}{n}\bar{u}_n(x) \\ \frac{r}{n}\bar{p}_n(x) \\ \frac{Et'r^2}{n^2}\bar{v}_n(x) \end{Vmatrix} = \begin{bmatrix} \cosh P_{1n}x & \cosh P_{2n}x & \sinh P_{1n}x & \sinh P_{2n}x \\ \frac{1}{P_{1n}^2}\cosh P_{1n}x & \frac{1}{P_{2n}^2}\cosh P_{2n}x & \frac{1}{P_{1n}^2}\sinh P_{1n}x & \frac{1}{P_{2n}^2}\sinh P_{2n}x \\ \frac{1}{P_{1n}}\sinh P_{1n}x & \frac{1}{P_{2n}}\sinh P_{2n}x & \frac{1}{P_{1n}}\cosh P_{1n}x & \frac{1}{P_{2n}}\cosh P_{2n}x \\ \frac{\sigma_{1n}}{P_{1n}}\sinh P_{1n}x & \frac{\sigma_{2n}}{P_{2n}}\sinh P_{2n}x & \frac{\sigma_{1n}}{P_{1n}}\cosh P_{1n}x & \frac{\sigma_{2n}}{P_{2n}}\cosh P_{2n}x \end{bmatrix} \begin{Vmatrix} N_1 \\ J_1 \\ N_2 \\ J_2 \end{Vmatrix} \tag{39}$$

The 4 x 4 matrix appearing in equation (39) can be factored into two matrices, each of which is conveniently written in partitioned form:

$$\begin{Bmatrix} \bar{q}_n(x) \\ \frac{Et'r}{n}\bar{u}_n(x) \\ \frac{r}{n}\bar{p}_n(x) \\ \frac{Et'r^2}{n^2}\bar{v}_n(x) \end{Bmatrix} = \begin{bmatrix} [\lambda] & 0 \\ 0 & [\Omega] \end{bmatrix} \begin{bmatrix} [C_n(x)] & [S_n(x)] \\ [S_n(x)] & [C_n(x)] \end{bmatrix} \begin{Bmatrix} N_1 \\ J_1 \\ N_2 \\ J_2 \end{Bmatrix} \qquad (40)$$

where

$$[\lambda] = \begin{bmatrix} 1 & 1 \\ \frac{1}{P_{1n}^2} & \frac{1}{P_{2n}^2} \end{bmatrix}; \quad [\Omega] = \begin{bmatrix} \frac{1}{P_{1n}} & \frac{1}{P_{2n}} \\ \frac{\sigma_{1n}}{P_{1n}} & \frac{\sigma_{2n}}{P_{2n}} \end{bmatrix}$$

$$[C_n(x)] = \begin{bmatrix} \cosh P_{1n}x & 0 \\ 0 & \cosh P_{2n}x \end{bmatrix}; \quad [S_n(x)] = \begin{bmatrix} \sinh P_{1n}x & 0 \\ 0 & \sinh P_{2n}x \end{bmatrix} \qquad (41)$$

The four undetermined coefficients are evaluated in terms of $\bar{q}_n(o)$, $\bar{u}_n(o)$, $\bar{p}_n(o)$, and $\bar{v}_n(o)$ by setting $x = 0$ in equation (40). For $x = 0$, the hyperbolic sine is zero and the hyperbolic cosine is unity. Hence,

$$\begin{Bmatrix} \bar{q}_n(o) \\ \frac{Et'r}{n}\bar{u}_n(o) \end{Bmatrix} = [\lambda] \begin{Bmatrix} N_1 \\ J_1 \end{Bmatrix} \qquad (42)$$

$$\begin{Bmatrix} \frac{r}{n}\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\bar{v}_n(o) \end{Bmatrix} = [\Omega] \begin{Bmatrix} N_2 \\ J_2 \end{Bmatrix} \qquad (43)$$

Invert these equations and write

$$\left|\begin{array}{c} N_1 \\ J_1 \\ N_2 \\ J_2 \end{array}\right| = \begin{bmatrix} [\lambda]^{-1} & 0 \\ 0 & [\Omega]^{-1} \end{bmatrix} \left|\begin{array}{c} \bar{q}_n(o) \\ \frac{Et'r}{n}\bar{u}_n(o) \\ \frac{r}{n}\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\bar{v}_n(o) \end{array}\right| \tag{44}$$

$[\lambda]^{-1}$ and $[\Omega]^{-1}$ are easily evaluated using the definitions of σ_{1n} and σ_{2n}:

$$[\lambda]^{-1} = \frac{P_{1n}^2 P_{2n}^2}{P_{1n}^2 - P_{2n}^2} \left[\begin{array}{c|c} \frac{1}{P_{2n}^2} & -1 \\ \hline -\frac{1}{P_{1n}^2} & +1 \end{array}\right] \tag{45}$$

$$[\Omega]^{-1} = \frac{P_{1n}^2 \quad P_{2n}^2}{P_{1n}^2 - P_{2n}^2} \left[\begin{array}{c|c} P_{1n}\sigma_{2n} & -P_{1n} \\ \hline -P_{2n}\sigma_{1n} & +P_{2n} \end{array}\right] \tag{46}$$

The complete result for transmission coefficients is obtained by substituting from equation (44) into equation (40):

$$\left|\begin{array}{c} \bar{q}_n(x) \\ \frac{Et'r}{n}\bar{u}_n(x) \\ \frac{r}{n}\bar{p}_n(x) \\ \frac{Et'r^2}{n^2}\bar{v}_n(x) \end{array}\right| = \begin{bmatrix} [\lambda][C_n(x)][\lambda]^{-1} & [\lambda][S_n(x)][\Omega]^{-1} \\ [\Omega][S_n(x)][\lambda]^{-1} & [\Omega][C_n(x)][\Omega]^{-1} \end{bmatrix} \left|\begin{array}{c} \bar{q}_n(o) \\ \frac{Et'r}{n}\bar{u}_n(o) \\ \frac{r}{n}\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\bar{v}_n(o) \end{array}\right| \tag{47}$$

Evaluation of Transmission Coefficients for Real Roots

The roots of the characteristic equation are all real for $n \geq N_c$. In this case, P_{1n} and P_{2n} may be replaced by α_{1n} and α_{2n} in the results of the preceding section. The terms in the square matrix on the right side of equation (47) are easily evaluated by carrying out the indicated operations.

The terminology of equation (19) is used in identifying coefficients, and the identities of equation (26) are employed to reduce the number of calculations.

For real roots, $n \geq N_c$

$$T_1 = T_{16} = K\left[\frac{1}{\alpha_{2n}^2}\cosh \alpha_{1n}x - \frac{1}{\alpha_{1n}^2}\cosh \alpha_{2n}x\right] \tag{48}$$

$$T_2 = T_{14} = K\left[-\cosh \alpha_{1n}x + \cosh \alpha_{2n}x\right] \tag{49}$$

$$T_3 = T_{15} = \frac{K}{\alpha_{1n}^2\alpha_{2n}^2}\left[\cosh \alpha_{1n}x - \cosh \alpha_{2n}x\right] \tag{50}$$

$$T_4 = T_{13} = K\left[-\frac{1}{\alpha_{1n}^2}\cosh \alpha_{1n}x + \frac{1}{\alpha_{2n}^2}\cosh \alpha_{2n}x\right] \tag{51}$$

$$T_5 = T_8 = T_{10} = K\left[-\frac{1}{\alpha_{1n}}\sinh \alpha_{1n}x + \frac{1}{\alpha_{2n}}\sinh \alpha_{2n}x\right] \tag{52}$$

$$T_6 = K\left[-\alpha_{1n}\sinh \alpha_{1n}x + \alpha_{2n}\sinh \alpha_{2n}x\right] \tag{53}$$

$$T_7 = K\left[\frac{\sigma_{2n}}{\alpha_{1n}}\sinh \alpha_{1n}x - \frac{\sigma_{1n}}{\alpha_{2n}}\sinh \alpha_{2n}x\right] \tag{54}$$

$$T_9 = T_{12} = \frac{K}{\alpha_{1n}^2\alpha_{2n}^2}\left[\alpha_{1n}\sinh \alpha_{1n}x - \alpha_{2n}\sinh \alpha_{2n}x\right] \tag{55}$$

$$T_{11} = \frac{K}{\alpha_{1n}^2\alpha_{2n}^2}\left[\alpha_{1n}\sigma_{1n}\sinh \alpha_{1n}x - \alpha_{2n}\sigma_{2n}\sinh \alpha_{2n}x\right] \tag{56}$$

K, α_{1n}, α_{2n}, σ_{1n}, and σ_{2n} are expressed in terms of the shell quantities, as follows: α_{1n} and α_{2n} are given in equation (32) of reference 1.

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$$\left.\begin{matrix}\alpha_{1n}\\ \alpha_{2n}\end{matrix}\right\} = \frac{1}{L_c} \cdot \frac{n\sqrt{n^2-1}}{\sqrt{6}} \left[\frac{n^2-1}{3}\left(\frac{L_r}{L_c}\right)^2 \pm \sqrt{\frac{(n^2-1)^2}{9}\left(\frac{L_r}{L_c}\right)^4 - 1}\ \right]^{\frac{1}{2}} \tag{57}$$

σ_{1n} and σ_{2n} defined in equation (34) are, by virtue of the functional form of α_{1n} and α_{2n}:

$$\sigma_{1n} = \frac{1}{\alpha_{1n}^2} - \frac{4L_r^2}{n^2} = -\frac{1}{\alpha_{2n}^2} \qquad \text{and} \qquad \sigma_{2n} = \frac{1}{\alpha_{2n}^2} - \frac{4L_r^2}{n^2} = -\frac{1}{\alpha_{1n}^2}$$

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$$K = \frac{\alpha_{1n}^2\,\alpha_{2n}^2}{\alpha_{1n}^2 - \alpha_{2n}^2} = \frac{n^2(n^2-1)}{12\,L_c^2} \cdot \frac{1}{\sqrt{a_n^2 - 1}} \tag{58}$$

and

$$\frac{K}{\alpha_{1n}^2\,\alpha_{2n}^2} = \frac{3\,L_c^2}{n^2(n^2-1)a_n}$$

Evaluation of Transmission Coefficients for Complex Roots

The roots of the characteristic equation are complex for $n < N_c$.

In this case:

$$P_{1n} = \alpha_n + j\beta_n$$

and

$$P_{2n} = \alpha_n - j\beta_n = P_{1n}^* \tag{59}$$

where P_{1n}^* means the complex conjugate of P_{1n}.

Equations (48) through (56) of the previous section are valid with the substitution, $\alpha_{1n} = \alpha_n + j\beta_n$ and $\alpha_{2n} = \alpha_n - j\beta_n$, because no special properties of the real roots were employed. If this substitution is made, it is observed that the second term inside the bracket in the formula for each coefficient is the conjugate of the first term.

Let Γ be the value of the first term. Then the value of the bracketed expression is

$$\Gamma - \Gamma^* = 2\,j\,\mathrm{Im}\cdot\Gamma \tag{60}$$

where $\mathrm{Im}\cdot\Gamma$ is the imaginary part of Γ.

The factor K has the following value:

$$K = \frac{P_{1n}^2\,P_{2n}^2}{P_{1n}^2 - P_{2n}^2} = \frac{(\alpha_n^2 + \beta_n^2)^2}{4\,j\,\alpha_n\beta_n} \tag{61}$$

Hence,

$$K(\Gamma - \Gamma^*) = \frac{(\alpha_n^2 + \beta_n^2)^2}{2\,\alpha_n\beta_n}\;\mathrm{Im}\cdot\Gamma \tag{62}$$

and

$$\frac{K(\Gamma - \Gamma^*)}{P_{1n}^2\;P_{2n}^2} = \frac{\mathrm{Im}\cdot\Gamma}{2\,\alpha_n\beta_n} \tag{63}$$

The hyperbolic functions become:

$$\cosh\,(\alpha_n + j\beta_n)x = \cosh\,\alpha_n x\,\cos\,\beta_n x + j\,\sinh\,\alpha_n x\,\sin\,\beta_n x \tag{64}$$

$$\sinh\,(\alpha_n + j\beta_n)x = \sinh\,\alpha_n x\,\cos\,\beta_n x + j\,\cosh\,\alpha_n x\,\sin\,\beta_n x \tag{65}$$

To simplify the writing of the equations, use the following abbreviations:

$$\left.\begin{aligned} f_{1n}(x) &= \cosh\,\alpha_n x\,\cos\,\beta_n x; & f_{2n}(x) &= \sinh\,\alpha_n x\,\sin\,\beta_n x \\ f_{3n}(x) &= \sinh\,\alpha_n x\,\cos\,\beta_n x; & f_{4n}(x) &= \cosh\,\alpha_n x\,\sin\,\beta_n x \end{aligned}\right. \tag{66}$$

The factors $\mathrm{Im}\,\Gamma$ are evaluated below for each of the terms in the transmission matrix.

$$\left.\begin{aligned} T_1:\quad \Gamma &= \frac{(f_{1n} + j\,f_{2n})}{(\alpha_n - j\beta_n)^2} = \frac{(\alpha_n^2 - \beta_n^2 + 2\,j\,\alpha_n\beta_n)}{(\alpha_n^2 + \beta_n^2)^2}(f_{1n} + j\,f_{2n}) \\ \mathrm{Im}\cdot\Gamma &= \frac{1}{(\alpha_n^2 + \beta_n^2)^2}\left[\,2\alpha_n\beta_n\,f_{1n} + (\alpha_n^2 - \beta_n^2)\,f_{2n}\,\right] \end{aligned}\right\} \tag{67}$$

$$
T_2: \quad \left.\begin{aligned} \Gamma &= -(f_{1n} + j f_{2n}) \\ I m \cdot \Gamma &= -f_{2n} \end{aligned}\right\} \tag{68}
$$

$$
T_3: \quad \left.\begin{aligned} \Gamma &= f_{1n} + j f_{2n} \\ I m \cdot \Gamma &= f_{2n} \end{aligned}\right\} \tag{69}
$$

$$
T_4: \quad \left.\begin{aligned} \Gamma &= \frac{-1}{(\alpha_n + j\beta_n)^2} (f_{1n} + j f_{2n}) \\ I m \cdot \Gamma &= \frac{1}{(\alpha_n^2 + \beta_n^2)^2} \left[2\alpha_n\beta_n f_{1n} - (\alpha_n^2 - \beta_n^2) f_{2n} \right] \end{aligned}\right\} \tag{70}
$$

$$
T_8: \quad \left.\begin{aligned} \Gamma &= -\frac{(f_{3n} + j f_{4n})}{(\alpha_n + j\beta_n)} \\ I m \cdot \Gamma &= \left(\beta_n f_{3n} - \alpha_n f_{4n}\right)/(\alpha_n^2 + \beta_n^2) \end{aligned}\right\} \tag{71}
$$

$$
T_6: \quad \left.\begin{aligned} \Gamma &= -(\alpha_n + j\beta_n)(f_{3n} + j f_{4n}) \\ I m \cdot \Gamma &= -\beta_n f_{3n} - \alpha_n f_{4n} \end{aligned}\right\} \tag{72}
$$

$$
T_7: \quad \Gamma = \frac{\sigma_{2n}}{(\alpha_n + j\beta_n)} (f_{3n} + j f_{4n})
$$

Now,

$$
\sigma_{2n} = \frac{1}{(\alpha_n - j\beta_n)^2} - \frac{4 L_r^2}{n^2} = \frac{\alpha_n^2 - \beta_n^2}{(\alpha_n^2 + \beta_n^2)^2} - \frac{4 L_r^2}{n^2} + j \frac{2\alpha_n \beta_n}{(\alpha_n^2 + \beta_n^2)^2} \tag{73}
$$

The equations that follow can be simplified by writing:

$$\sigma_{2n} = \mathrm{Re}\cdot\sigma - j\,\mathrm{Im}\cdot\sigma \qquad (74)$$
$$\sigma_{1n} = \mathrm{Re}\cdot\sigma + j\,\mathrm{Im}\cdot\sigma$$

$$\therefore \quad \Gamma = \frac{(\mathrm{Re}\cdot\sigma - j\,\mathrm{Im}\cdot\sigma)(\alpha_n - j\beta_n)}{\alpha_n^2 + \beta_n^2}\,(f_{3n} + j\,f_{4n})$$

$$\mathrm{Im}\cdot\Gamma = \frac{-f_{3n}(\beta_n\mathrm{Re}\cdot\sigma + \alpha_n\mathrm{Im}\cdot\sigma) + f_{4n}(\alpha_n\,\mathrm{Re}\cdot\sigma - \beta_n\,\mathrm{Im}\cdot\sigma)}{\alpha_n^2 + \beta_n^2} \qquad (75)$$

$$T_9: \quad \Gamma = (\alpha_n + j\,\beta_n)(f_{3n} + j\,f_{4n})$$

$$\mathrm{Im}\cdot\Gamma = \beta_n f_{3n} + \alpha_n f_{4n} \qquad (76)$$

$$T_{11}: \quad \Gamma = (\alpha_n + j\beta_n)(\mathrm{Re}\cdot\sigma + j\;\mathrm{Im}\cdot\sigma)(f_{3n} + j\,f_{4n})$$

$$\mathrm{Im}\cdot\Gamma = f_{3n}(\alpha_n\,\mathrm{Im}\cdot\sigma + \beta_n\,\mathrm{Re}\cdot\sigma) + f_{4n}(\alpha_n\,\mathrm{Re}\cdot\sigma - \beta_n\,\mathrm{Im}\cdot\sigma) \qquad (77)$$

Hence, the transmission coefficients for complex roots are:

$$T_1 = T_{16} = f_{1n} + \frac{(\alpha_n^2 - \beta_n^2)}{2\alpha_n\beta_n}\,f_{2n} \qquad (78)$$

$$T_2 = T_{14} = \frac{-(\alpha_n^2 + \beta_n^2)^2}{2\,\alpha_n\beta_n}\,f_{2n} \qquad (79)$$

$$T_3 = T_{15} = \frac{1}{2\,\alpha_n\beta_n}\,f_{2n} \qquad (80)$$

$$T_4 = T_{13} = f_{1n} - \frac{(\alpha_n^2 - \beta_n^2)}{2\,\alpha_n\beta_n}\,f_{2n} \qquad (81)$$

$$T_5 = T_8 = T_{10} = \frac{(\alpha_n^2 + \beta_n^2)}{2\,\alpha_n\beta_n}\left[\,\beta_n f_{3n} - \alpha_n f_{4n}\,\right] \qquad (82)$$

$$T_6 = \frac{(\alpha_n^2 + \beta_n^2)^2}{2\alpha_n \beta_n} \left[-\beta_n f_{3n} - \alpha_n f_{4n} \right] \tag{83}$$

$$T_7 = \frac{(\alpha_n^2 + \beta_n^2)}{2\alpha_n \beta_n} \left[-(\alpha_n \operatorname{Im}\sigma + \beta_n \operatorname{Re}\sigma) f_{3n} + (\alpha_n \operatorname{Re}\sigma - \beta_n \operatorname{Im}\sigma) f_{4n} \right] \tag{84}$$

$$T_9 = T_{12} = \frac{1}{2\alpha_n \beta_n} \left[\beta_n f_{3n} + \alpha_n f_{4n} \right] \tag{85}$$

$$T_{11} = \frac{1}{2\alpha_n \beta_n} \left[(\alpha_n \operatorname{Im}\sigma + \beta_n \operatorname{Re}\sigma) f_{3n} + (\alpha_n \operatorname{Re}\sigma - \beta_n \operatorname{Im}\sigma) f_{4n} \right] \tag{86}$$

The function of α_n and β_n appearing in these equations are evaluated below in terms of n, L_r, L_c and a_n. α_n and β_n are given in equations (29) and (30) of reference 1, as:

$$\left.\begin{matrix}\alpha_n \\ \beta_n\end{matrix}\right\} = \frac{1}{L_c} \cdot \frac{n\sqrt{n^2 - 1}}{2\sqrt{3}} \sqrt{1 \pm a_n}$$

$$\operatorname{Re}\sigma = -\frac{2 L_r^2}{n^2} \tag{87}$$

$$\operatorname{Im}\sigma = -\frac{6 L_c^2 \sqrt{1 - a_n^2}}{n^2 (n^2 - 1)} \tag{88}$$

$$\frac{\alpha_n^2 - \beta_n^2}{2\alpha_n \beta_n} = \frac{a_n}{\sqrt{1 - a_n^2}} \tag{89}$$

$$\frac{\alpha_n^2 + \beta_n^2}{2\alpha_n \beta_n} = \frac{1}{\sqrt{1 - a_n^2}} \tag{90}$$

$$\frac{\left(\alpha_n^2 + \beta_n^2\right)^2}{2\alpha_n \beta_n} = \frac{n^2 (n^2 - 1)}{6 L_c^2} \cdot \frac{1}{\sqrt{1 - a_n^2}} \tag{91}$$

$$\frac{1}{2\alpha_n \beta_n} = \frac{6 L_c^2}{n^2 (n^2 - 1)} \cdot \frac{1}{\sqrt{1 - a_n^2}} \tag{92}$$

f_{1n}, f_{2n}, f_{3n} and f_{4n} are defined in equation (66).

Using the equations derived, the transmission coefficients can be computed for each value of the harmonic index, n . This has been done for $n = 2$ and $L_r/L_c = 0.4$. The results are shown in figure 5. It remains to be shown, by means of a number of simple but useful examples, how these coefficients are used in the solution of practical problems.

EXAMPLES OF DISCONTINUITIES NEAR THE LOADED FRAME

Discussion

In the first section the foundations are laid for solving shell problems involving certain discontinuities in the shell. It is shown there that such problems can be approximately solved by use of the tables of reference 2, once the function f(2) is known. The limitations of this simplification are discussed in the first section. While the major effort is devoted to the case of the harmonic index equal to 2, it is desirable to indicate the exact solutions at the same time. These consist of a superposition of the stress systems for each n. In practice, adequate convergence of the Fourier series is obtained using only a few values of n . For this reason the example problems are solved in terms of the harmonic index, n , and specialized to $n = 2$ for the numerical calculations that yield the approximate solution.

In this section solutions are derived in the form of matrix equations, giving the input impedance for a variety of problems involving discontinuities near the loaded frame. It is implicitly assumed in the derivation of the tables of reference 2 that the shell is symmetric about the loaded frame and therefore $\bar{q}_n(o+) = \bar{q}_n(o-) = \Delta\bar{q}_n/2$. If the loaded frame is at a free end, either $\bar{q}_n(o^-)$ or $\bar{q}_n(+)$ equals $\Delta\bar{q}_n$. For any condition between these extremes the ratio of $\bar{q}_n(o)$ to $\Delta\bar{q}_n$ depends on the problem and is a function of n . Hence, when the approximate solution involving the modification of γ and use of the tables of reference 2 is utilized, the loads per inch in the shell may be in error when the shell and frame stiffness is not symmetrical about the loaded frame. In the following text the results for symmetric and antisymmetric externally applied loadings are identical and only the symmetric case is considered. In all cases the shell elements are taken to have a circumferential-bending stiffness per unit length, due to "smearing out" all the frames except those specifically mentioned as producing the discontinuity and the externally loaded frame.

Two Rigid Bulkheads Symmetrically Placed About the Loaded Frame

The loaded frame is at $x = 0$; the shell extends to ∞ in both directions and has infinitely stiff bulkheads at $x = \pm \ell$ (figure 6). The transmission matrix giving the relationships between force and displacement harmonic coefficients for the finite shell lengths is given in equation (19). The transmission coefficients are given in the second section for $n > N_c$, and $n < N_c$.

Relationships between the force and displacement coefficients at the end of the semi-infinite shell are given in the second section (equation [27]) .

The equations of connection of rigid bulkheads are taken from the second section and are as follows:

$$\left.\begin{array}{lll} \bar{p}_n(\ell^*) = \bar{p}_n(\ell) & ; & \bar{u}_n(\ell^*) = \bar{u}_n(\ell) \\ \bar{q}_n(\ell^*) = \Delta\bar{q}_n(\ell) + \bar{q}_n(\ell) & ; & \bar{v}_n(\ell^*) = \bar{v}_n(\ell) \end{array}\right\} \quad (93)$$

Boundary Conditions

Due to symmetry, it is necessary to consider only one side of the shell. For the same reason:

$$\bar{u}_n(o) = 0 \quad (94)$$

The bulkhead, rigid in its own plane, prevents tangential displacements, i.e.:

$$\bar{v}_n(\ell^*) = \bar{v}_n(\ell) = 0 \quad (95)$$

Using equation (95) in equation (27a) we have:

$$\bar{u}_n(\ell^*) = \left[Z_{22} - \frac{Z_{12}\,Z_{21}}{Z_{11}} \right] \bar{p}_n(\ell^*) \equiv Z\,\bar{p}_n(\ell^*) \quad (96)$$

From equation (93) it is seen that equation (96) can be written as:

$$\bar{u}_n(\ell) = Z\,\bar{p}_n(\ell) \quad (97)$$

Solution

Substituting equations (94), (95) and (97) into equation (19) we obtain the following matrix equation:

$$\begin{vmatrix} \bar{q}_n(\ell) \\ \frac{Et'r}{n} \cdot Z \cdot \bar{p}_n(\ell) \\ \frac{r}{n}\bar{p}_n(\ell) \\ 0 \end{vmatrix} = \begin{bmatrix} T_1 & T_2 & T_5 & T_6 \\ T_3 & T_4 & T_7 & T_8 \\ T_9 & T_{10} & T_{13} & T_{14} \\ T_{11} & T_{12} & T_{15} & T_{16} \end{bmatrix} \begin{vmatrix} \bar{q}_n(o) \\ 0 \\ \frac{r}{n}\,\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\,\bar{v}_n(o) \end{vmatrix} \quad (98)$$

Multiplying the third row by Et'Z and subtracting it from the second, thus eliminating the third row, equation (98) is simplified to:

$$\begin{vmatrix} \bar{q}_n(\ell) \\ 0 \\ 0 \end{vmatrix} = \begin{bmatrix} T_1 & T_5 & T_6 \\ T_3 - Et'Z\,T_9 & T_7 - Et'Z\,T_{13} & T_8 - Et'Z\,T_{14} \\ T_{11} & T_{15} & T_{16} \end{bmatrix} \begin{vmatrix} \bar{q}_n(o) \\ \frac{r}{n}\,\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\bar{v}_n(o) \end{vmatrix} \quad (99)$$

Inverting this equation we obtain:

$$\begin{vmatrix} \bar{q}_n(o) \\ \frac{r}{n}\,\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\bar{v}_n(o) \end{vmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{vmatrix} \bar{q}_n(\ell) \\ 0 \\ 0 \end{vmatrix} \quad (100)$$

In any case of symmetry about the loaded frame,

$$\bar{q}_n(o) = \frac{\overline{\Delta q_n}}{2}$$

Therefore, the Input Impedance $= \dfrac{\bar{v}_n(o)}{\overline{\Delta q}_n(o)} = \dfrac{n^2}{2\,Et'r^2} \cdot \dfrac{A_{31}}{A_{11}}$ (101)

where

$$A_{11} = \left[T_{16}(T_7 - Et'Z\,T_{13}) - T_{15}(T_8 - Et'Z\,T_{14}) \right] \Big/ \,|\,T\,|$$

$$A_{31} = \left[T_{15}(T_3 - Et'Z\,T_9) - T_{11}(T_7 - Et'Z\,T_{13}) \right] \Big/ \,|\,T\,|$$

and $|\,T\,|$ is the determinant of the square matrix in equations (99). Clearly, it is not necessary to evaluate this determinant for the solution of this problem.

Substitution of Input Impedance given by equation (101) into equation (4) gives f(n). Figure 10 shows f(2) plotted as a function of ℓ/L_c for $L_r/L_c = 0.4$. As a check on the basic premise of the first section that f(n) can be replaced by f(2), i.e., $f(n)/f(2) \doteq 1.0$, it can be easily shown that for this problem, $f(2)/f(3) = 0.70$ for $\ell/L_c = 1.0$. Satisfactory agreement was obtained for the following problems as well.

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A Rigid Bulkhead on One Side of the Loaded Frame

The shell, frames and coordinate system are shown in figure 7. Since there is no symmetry about the loaded frame, $\bar{q}_n(o) \neq \overline{\Delta q}_n/2$ and the shell on both sides of the loaded frame must be taken into account.

Therefore, there are three elements of shell governed by three sets of equations, as given below, to be considered.

For the finite length, $0 < x < \ell$: equation (19) holds.

For semi-infinite shell extending from ℓ to ∞ ,

$$\left| \begin{matrix} \bar{v}_n(\bar{\ell}) \\ \bar{u}_n(\bar{\ell}) \end{matrix} \right| = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \left| \begin{matrix} \bar{q}_n(\bar{\ell}) \\ \bar{p}_n(\bar{\ell}) \end{matrix} \right| \tag{102}$$

In the case of the semi-infinite shell from $x^* = 0$ to ∞ ,

$$\left| \begin{matrix} \bar{v}_n(o^*) \\ \bar{u}_n(o^*) \end{matrix} \right| = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \left| \begin{matrix} \bar{q}_n(o^*) \\ \bar{p}_n(o^*) \end{matrix} \right| \tag{103}$$

Boundary Conditions

In the second section, the boundary conditions relating the harmonic coefficients are discussed and the results are quoted here:

At $x = x^* = 0$

$$\left. \begin{matrix} \bar{p}_n(o^*) = \bar{p}_n(o) & ; & \bar{u}_n(o^*) = -\bar{u}_n(o) \\ \bar{v}_n(o^*) = \bar{v}_n(o) & ; & \bar{q}_n(o^*) = \overline{\Delta q}_n(o) - \bar{q}_n(o) \end{matrix} \right\} \tag{104}$$

At $x = \ell$

$$\left. \begin{matrix} \bar{p}_n(\bar{\ell}) = \bar{p}_n(\ell) & ; & \bar{u}_n(\ell) = \bar{u}_n(\ell) \\ \bar{v}_n(\bar{\ell}) = \bar{v}_n(\ell) & ; & \bar{q}_n(\ell) = \overline{\Delta q}_n(\ell) + \bar{q}_n(\ell) \end{matrix} \right\} \tag{105}$$

The rigid bulkhead prevents tangential displacements, i.e.,

$$\bar{v}_n(\ell) = \bar{v}_n(\ell) = 0 \tag{106}$$

Solution

Using boundary condition of equation (106) in equations (102) allows $\bar{u}_n(\ell)$ to be expressed in terms of $\bar{p}_n(\ell)$. Substituting equations (105) into this expression gives:

$$\bar{u}_n(\ell) = \left[Z_{22} - \frac{Z_{12}\, Z_{21}}{Z_{11}} \right] \bar{p}_n(\ell) \tag{107}$$

Substituting equation (107) into equation (19), multiplying the third row by Et'Z and subtracting from the second, we have :

$$\begin{Bmatrix} \bar{q}_n(\ell) \\ 0 \\ \frac{r}{n}\,\bar{p}_n(\ell) \\ 0 \end{Bmatrix} = \begin{bmatrix} T_1 & T_2 & T_5 & T_6 \\ T_3 - Et'Z\,T_9 & T_4 - Et'Z\,T_{10} & T_7 - Et'Z\,T_{13} & T_8 - Et'Z\,T_{14} \\ T_9 & T_{10} & T_{13} & T_{14} \\ T_{11} & T_{12} & T_{15} & T_{16} \end{bmatrix} \begin{Bmatrix} \bar{q}_n(o) \\ \frac{Et'r}{n}\,\bar{u}_n(o) \\ \frac{r}{n}\,\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\,\bar{v}_n(o) \end{Bmatrix} \tag{108}$$

Apply boundary condition (104) to equations (103), and rearrange the latter as:

$$\begin{Bmatrix} \bar{u}_n(o) \\ \bar{p}_n(o) \end{Bmatrix} = \begin{bmatrix} Z_{21} - \frac{Z_{11} Z_{22}}{Z_{12}} & -\frac{Z_{22}}{Z_{11}} \\ \frac{Z_{11}}{Z_{12}} & \frac{1}{Z_{12}} \end{bmatrix} \begin{Bmatrix} \bar{q}_n(o) - \Delta\bar{q}_n(o) \\ \bar{v}_n(o) \end{Bmatrix} \tag{109}$$

By substituting equation (109) into equation (108), $\bar{u}_n(o)$ and $\bar{p}_n(o)$ are eliminated and the right-hand-column matrix will contain only $\bar{q}_n(o)$, $\Delta\bar{q}_n(o)$, and $Et'r^2\,\bar{v}_n(o) / n^2$. If, at the same time, the third row of equation (108) is deleted, the matrix equation is simplified to:

$$\begin{Bmatrix} \bar{q}_n(\ell) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{Bmatrix} \bar{q}_n(o) \\ \Delta\bar{q}_n(o) \\ \frac{Et'r^2}{n^2}\,\bar{v}_n(o) \end{Bmatrix} \tag{110}$$

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where

$$H_{11} = \left[T_1 + \frac{Etr}{n} \left\{ Z_{21} - \frac{Z_{11} Z_{22}}{Z_{12}} \right\} T_2 + \frac{r Z_{11}}{n Z_{12}} \; T_5 \right]$$

$$H_{12} = \left[- \frac{Et'r}{n} \left\{ Z_{21} - \frac{Z_{11} Z_{22}}{Z_{12}} \right\} T_2 - \frac{r Z_{11}}{n Z_{12}} \; T_5 \right]$$

$$H_{13} = \left[T_6 - \frac{nZ_{22}}{rZ_{12}} T_2 + \frac{n}{Et'r} \; \frac{T_5}{Z_{12}} \right]$$

$$H_{21} = \left[(T_3 - Et'Z\,T_9) + \frac{Et'r}{n} \left\{ Z_{21} - \frac{Z_{11} Z_{22}}{Z_{12}} \right\} (T_4 - Et'ZT_{10}) + \frac{rZ_{11}}{nZ_{12}} (T_7 - Et'ZT_{13}) \right]$$

$$H_{22} = \left[- \frac{Et'r}{n} \left\{ Z_{21} - \frac{Z_{11} Z_{22}}{Z_{12}} \right\} (T_4 - Et'ZT_{10}) - \frac{rZ_{11}}{nZ_{12}} (T_7 - Et'ZT_{13}) \right]$$

$$H_{23} = \left[(T_8 - Et'ZT_{14}) - (T_4 - Et'ZT_{10}) \frac{nZ_{22}}{rZ_{11}} + \frac{n}{Et'rZ_{12}} (T_7 - Et'ZT_{13}) \right]$$

$$H_{31} = \left[T_{11} + \frac{Et'r}{n} \left\{ Z_{21} - \frac{Z_{11} Z_{22}}{Z_{12}} \right\} T_{12} + \frac{rZ_{11}}{nZ_{12}} \; T_{15} \right]$$

$$H_{32} = \left[- \frac{Et'r}{n} \left\{ Z_{21} - \frac{Z_{11} Z_{22}}{Z_{12}} \right\} T_{12} - \frac{rZ_{11}}{nZ_{12}} \; T_{15} \right]$$

$$H_{33} = \left[T_{16} - \frac{nZ_{22}}{rZ_{12}} \; \Gamma_{12} + \frac{n}{Et'rZ_{12}} \; T_{15} \right]$$

The inversion of equation (110) yields the solution for the Input Impedance as follows:

$$\frac{\bar{v}_n(o)}{\Delta \bar{q}_n(o)} = \frac{-n^2}{Et'r^2} \; \frac{\left\{ H_{21} H_{32} - H_{22} H_{31} \right\}}{\left\{ H_{21} H_{33} - H_{23} H_{31} \right\}} \qquad (111)$$

This equation substituted into equation (4) gives the $f(n)$ which is needed for the general solution of the shell problem. Figure 10 shows $f(2)$ as a function of ℓ/L_c for $L_r/L_c = 0.4$. (curve for $\gamma_\ell = \infty$)

A Heavy Frame on One Side of Loaded Frame

This case is similar to that of a single rigid bulkhead; the difference being that $\bar{v}_n(\ell)$, the tangential displacement at the heavy frame, is dependent on the stiffness of the frame. The shell, frames, and coordinate system are shown in figure 8.

Let I_ℓ be the inertia of frame at $x = \ell$. For a frame that is not subjected to any external loads, we have from equation (62) of reference 1,

$$\bar{v}_n(\ell) = - \frac{r^4}{EI_\ell(n^3 - n)^2} \overline{\Delta q}_n(\ell) \tag{112}$$

From the definitions of L_c and γ_ℓ we have,

$$\frac{r^4}{I_\ell} = \frac{36\, L_c^4}{2\gamma_\ell t' r^2} \tag{113}$$

Substituting equation (113) into equation (112),

$$\bar{v}_n(\ell) = \frac{-\,36}{(n^3 - n)^2} \cdot \frac{1}{2\gamma_\ell} \cdot \frac{L_c^3}{Et'r^2} \overline{\Delta q}_n(\ell) \equiv - W\overline{\Delta q}_n(\ell) \tag{114}$$

Substituting equation (114) into the stress-displacement relationships for the semi-infinite shell given in equation (102) and observing the equations of connection (105), equations (102) may be written:

$$\begin{vmatrix} - W\overline{\Delta q}_n(\ell) \\ \bar{u}_n(\ell) \end{vmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{vmatrix} \overline{\Delta q}_n(\ell) + \bar{q}_n(\ell) \\ \bar{p}_n(\ell) \end{vmatrix} \tag{115}$$

Using equation (114) and equation (115) enables equations for $\bar{v}_n(\ell)$ and $\bar{u}_n(\ell)$ to be written in terms of the stress harmonic coefficients, $\bar{q}_n(\ell)$ and $\bar{p}_n(\ell)$.

$$\begin{vmatrix} \bar{v}_n(\ell) \\ \bar{u}_n(\ell) \end{vmatrix} \; \begin{bmatrix} \dfrac{W\,Z_{11}}{(W + Z_{11})} & \dfrac{WZ_{12}}{(W + Z_{11})} \\ P^* & Z^* \end{bmatrix} \begin{vmatrix} \bar{q}_n(\ell) \\ \bar{p}_n(\ell) \end{vmatrix} \tag{116}$$

where

$$P^* = \left[Z_{21} - \frac{Z_{11}\, Z_{21}}{(W + Z_{11})} \right] \text{ and } Z^* = \left[Z_{22} - \frac{Z_{12}\, Z_{21}}{(W + Z_{11})} \right]$$

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As $I_\ell \rightarrow \infty$, $W \rightarrow 0$. Thus, it is obvious that, at the same time, $Z^* \rightarrow Z$, $P^* \rightarrow 0$, and $\bar{v}_n(\ell) \rightarrow 0$. In the limit we have the solution, given previously, for the rigid bulkhead.

Substituting for the displacement equation (116) into the transmission constant equation (19), we obtain

$$\left|\begin{matrix} \bar{q}_n(\ell) \\ \frac{Et'r}{n}\left\{P^*\bar{q}_n(\ell) + Z^*\bar{p}_n(\ell)\right\} \\ \frac{r}{n}\,\bar{p}_n(\ell) \\ \frac{Et'r^2}{n^2}\left\{\frac{WZ_{11}}{(W+Z_{11})}\,\bar{q}_n(\ell) + \frac{WZ_{12}}{(W+Z_{11})}\,\bar{p}_n(\ell)\right\} \end{matrix}\right| = \begin{bmatrix} T_1 & T_2 & T_5 & T_6 \\ T_3 & T_4 & T_7 & T_8 \\ T_9 & T_{10} & T_{13} & T_{14} \\ T_{11} & T_{12} & T_{15} & T_{16} \end{bmatrix} \left|\begin{matrix} \bar{q}_n(o) \\ \frac{Et'r}{n}\,\bar{u}_n(o) \\ \frac{r}{n}\,\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\,\bar{v}_n(o) \end{matrix}\right| \tag{117}$$

Multiply the first row by $\frac{Et'r^2}{n^2} \cdot \frac{WZ_{11}}{(W+Z_{11})} \equiv R$, the third row by $\frac{Et'r}{n} \cdot \frac{Z_{12}}{(W+Z_{11})} \equiv Q$, and subtract both from the fourth.

Then multiply the first row by $Et'rP/n$ and the third by $Et'Z^*$ and subtract from the second. By doing this two zeros are introduced into the left-hand-column matrix of equation (117) which becomes,

$$\left|\begin{matrix} \bar{q}_n(\ell) \\ 0 \\ \frac{r}{n}\bar{p}_n(\ell) \\ 0 \end{matrix}\right| = \begin{bmatrix} T_1 & T_2 & T_5 & T_6 \\ \begin{matrix}(T_3 - \frac{Et'r}{n}P^*T_1 \\ -Et'Z^*T_9)\end{matrix} & \begin{matrix}(T_4 - \frac{Et'r}{n}P^*T_2 \\ -EtZ^*T_{10})\end{matrix} & \begin{matrix}(T_7 - \frac{Et'r}{n}P^*T_5 \\ -Et'Z^*T_{13})\end{matrix} & \begin{matrix}(T_8 - \frac{Et'r}{n}P^*T_6 \\ -Et'Z^*T_{14})\end{matrix} \\ T_9 & T_{10} & T_{13} & T_{14} \\ T_{11}-RT_1-QT_9 & T_{12}-RT_2-QT_{10} & T_{15}-RT_5-QT_{13} & T_{16}-RT_6-QT_{14} \end{bmatrix} \left|\begin{matrix} \bar{q}_n(o) \\ \frac{Et'r}{n}\,\bar{u}_n(o) \\ \frac{r}{n}\,\bar{p}_n(o) \\ \frac{Et'r^2}{n^2}\bar{v}_n(o) \end{matrix}\right|$$

(118)

Note that except for the modified form of the square matrix, this equation is the same as equation (108), referring to the case of a rigid bulkhead. Hence, from this point on, the solution is the same as for the rigid bulkhead. Using the results previously obtained, we may write:

$$\begin{vmatrix} \bar{q}_n(\ell) \\ 0 \\ 0 \end{vmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{vmatrix} \bar{q}_n(o) \\ \overline{\Delta q}_n(o) \\ \frac{Et'r^2}{n^2} \ \bar{v}_n(o) \end{vmatrix} \tag{119}$$

where

$B_{11} = H_{11}$ of equation (110)

$B_{12} = H_{12}$ of equation (110)

$B_{13} = H_{13}$ of equation (110)

$$B_{21} = \left[\left(T_3 - \frac{Et'r}{n} P^*T_1 - Et'Z^*T_9\right) + \frac{Et'r}{n} \left\{ Z_{21} - \frac{Z_{11}Z_{22}}{Z_{12}} \right\} \left(T_4 - \frac{Et'r}{n} P^*T_2 - Et'Z^*T_{10}\right) + \frac{rZ_{11}}{nZ_{12}} \left(T_7 - \frac{Et'r}{n} P^*T_5 - Et'Z^*T_{13}\right)\right]$$

$$B_{22} = \left[- \frac{Et'r}{n} \left\{ Z_{21} - \frac{Z_{11}Z_{22}}{Z_{12}} \right\} \left(T_4 - \frac{Etr}{n} P^*T_2 - Et'Z^*T_{10}\right) - \frac{rZ_{11}}{nZ_{22}} \left(T_7 - \frac{Et'r}{n} P^*T_5 - Et'Z^*T_{13}\right) \right]$$

$$B_{23} = \left[\left(T_8 - \frac{Et'r}{n} P^*T_6 - Et'Z^*T_{14}\right) - \frac{nZ_{22}}{rZ_{12}} \left(T_4 - \frac{Et'r}{n} P^*T_2 - Et'Z^*T_{10}\right) + \frac{n}{Et'rZ_{12}} \left(T_7 - \frac{Et'r}{n} P^*T_5 - Et'Z^*T_{13}\right) \right]$$

$$B_{31} = \left[(T_{11} - RT_1 - QT_9) + \frac{Et'r}{n} \left\{ Z_{21} - \frac{Z_{11}Z_{22}}{Z_{12}} \right\} (T_{12} - RT_2 - QT_{10}) + \frac{rZ_{11}}{nZ_{12}} (T_{15} - RT_5 - QT_{13}) \right]$$

$$B_{32} = \left[-\frac{Et'r}{n} \left\{ Z_{21} - \frac{Z_{11}Z_{22}}{Z_{12}} \right\} (T_{12} - RT_2 - QT_{10}) - \frac{rZ_{11}}{nZ_{12}} (T_{15} - RT_5 - QT_{13}) \right]$$

$$B_{33} = \left[(T_{16} - RT_6 - QT_{14}) - \frac{nZ_{22}}{rZ_{12}} (T_{12} - RT_2 - QT_{10}) + \frac{n}{Et'rZ_{12}} (T_{15} - RT_5 - QT_{13}) \right]$$

The inversion of equations (119) gives the Input Impedance as

$$\begin{vmatrix} \bar{q}_n(o) \\ \Delta\bar{q}_n(o) \\ \frac{Et'r^2}{n^2}\bar{v}_n(o) \end{vmatrix} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix} \begin{vmatrix} \bar{q}_n(\ell) \\ 0 \\ 0 \end{vmatrix} \tag{120}$$

$$\therefore \quad \frac{\bar{v}_n(o)}{\Delta\bar{q}_n(o)} = \frac{n^2 V_{31}}{Et'r^2 V_{21}} = \frac{-n^2}{Et'r^2} \left\{ \frac{B_{21}B_{32} - B_{22}B_{31}}{B_{21}B_{33} - B_{23}B_{31}} \right\}$$

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The substitution of equation (120) into equation (4) gives the f(n) required for the general solution of the problem. Figure 10 shows f(2) for $L_r/L_c = 0.4$. and various values of γ_ℓ.

Free End at a Finite Distance From the Loaded Frame

Before finding the Input Impedance it is necessary to find the relationships between the stress and displacement harmonic coefficients for a finite length of shell, free at one end. The coordinate system is illustrated in figure 9.

The relationships between the harmonic coefficients at the ends of the finite length of shell are given by equation (19).

At the free end the stresses are zero:

$$\bar{p}_n(\ell) = \bar{q}_n(\ell) = 0 \tag{121}$$

Substituting these boundary conditions into equation (19) gives two equations for $\bar{v}_n(o)$ in terms of $\bar{p}_n(o)$, $\bar{q}_n(o)$, and $\bar{u}_n(o)$. Equating these values of $\bar{v}_n(o)$, an equation for $\bar{u}_n(o)$ in terms of stress coefficients $\bar{q}_n(o)$ and $\bar{p}_n(o)$ is obtained. A similar equation is given for $\bar{v}_n(o)$. These results are expressed as:

$$\begin{vmatrix} \bar{v}_n(o) \\ \bar{u}_n(o) \end{vmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{vmatrix} \bar{q}_n(o) \\ \bar{p}_n(o) \end{vmatrix} \tag{122}$$

where

$$X_{11} = \left[- \frac{T_1}{T_6} - \frac{T_2}{T_6} \frac{Et'r}{n} X_{21} \right] \frac{n^2}{Et'r^2}$$

$$X_{12} = \left[- \frac{T_5}{T_6} - \frac{T_2}{T_6} Et' X_{22} \right] \frac{n}{Et'r}$$

$$X_{21} = \frac{\left(\frac{T_9}{T_{14}} - \frac{T_1}{T_6}\right)}{\left(\frac{T_2}{T_6} - \frac{T_{10}}{T_{14}}\right)} \cdot \frac{n}{Et'r} \; ; \quad X_{22} = \frac{\left(\frac{T_{13}}{T_{14}} - \frac{T_5}{T_6}\right)}{\left(\frac{T_2}{T_6} - \frac{T_{10}}{T_{14}}\right)} \cdot \frac{1}{Et'}$$

It can be shown that $X_{12} = - X_{21}$. These results are used to find the Input Impedance.

Input Impedance

At the loaded frame the stress-displacement relationships for the finite length of shell are given by equation (122); for the semi-inifinite shell, they are given by equation (103).

The equations of connection or boundary conditions at the loaded frame are given by equation (104).

Applying equations (104) to (103) the latter may be rewritten as:

$$\begin{vmatrix} \bar{v}_n(o) \\ -\bar{u}_n(o) \end{vmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{vmatrix} \overline{\Delta q}_n(o) - \bar{q}_n(o) \\ \bar{p}_n(o) \end{vmatrix} \tag{123}$$

After combining equation (122) and (123) and rearranging, we have:

$$\begin{bmatrix} (X_{11} + Z_{11}) & (X_{12} - Z_{12}) \\ (X_{21} - Z_{21}) & (X_{22} + Z_{22}) \end{bmatrix} \begin{vmatrix} \bar{q}_n(o) \\ \bar{p}_n(o) \end{vmatrix} = \begin{vmatrix} Z_{11} \\ - Z_{21} \end{vmatrix} \begin{vmatrix} \overline{\Delta q}_n(o) \end{vmatrix} \tag{124}$$

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Inverting the square matrix to solve for $q_n(o)$ and $p_n(o)$, and substituting the result into equation (122) enables the Input Impedance to be given:

$$\frac{\bar{v}_n(o)}{\overline{\Delta q}_n(o)} = \begin{Bmatrix} X_{11} & X_{12} \end{Bmatrix} \begin{bmatrix} (X_{11} + Z_{11}) & (X_{12} - Z_{12}) \\ (X_{21} - Z_{21}) & (X_{11} + Z_{22}) \end{bmatrix}^{-1} \begin{vmatrix} Z_{11} \\ -Z_{21} \end{vmatrix} \tag{125}$$

It is known that when the free end is a large distance ($\ell/L_c > 1.72$) from the loaded frame, its effects become negligible and that X_{ik} equals Z_{ik} . This limiting case is now examined.

Equation (125) becomes

$$\frac{\bar{v}_n(o)}{\overline{\Delta q}_n(o)} = \begin{Bmatrix} Z_{11} & Z_{12} \end{Bmatrix} \begin{bmatrix} \frac{1}{2Z_{11}} & 0 \\ 0 & \frac{1}{2Z_{22}} \end{bmatrix} \begin{vmatrix} Z_{11} \\ -Z_{21} \end{vmatrix} \tag{126}$$

$$= 1/2 \left[Z_{11} - \frac{Z_{12} Z_{21}}{Z_{22}} \right]$$

This result can also be obtained by applying symmetry conditions $\overline{\Delta q}_n(o) = 2\bar{q}_n(o)$ and $\bar{u}_n(o) = 0$.

Built-In End at a Finite Distance From the Loaded Frame

All cases in which only a finite length of shell exists on one side of the loaded frame can be dealt with in the same manner. The boundary conditions at $x = \ell$ bring about the difference. Hence, the case of the free end will be taken as a basis for this and the two similar cases of a plane of symmetry or a plane of anti-symmetry at $x = \ell$.

Solution

Substituting the boundary conditions $\bar{u}_n(\ell) = \bar{v}_n(\ell) = 0$ into equations (19), the displacement-stress relationships for the finite shell length at the loaded frame are found as follows:

$$\begin{vmatrix} \bar{v}_n(o) \\ \bar{u}_n(o) \end{vmatrix} = \begin{bmatrix} X^*_{11} & X^*_{12} \\ X^*_{21} & X^*_{22} \end{bmatrix} \begin{vmatrix} \bar{q}_n(o) \\ \bar{p}_n(o) \end{vmatrix} \tag{127}$$

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where

$$X^*_{11} = \left[-\frac{T_3}{T_8} - \frac{T_4}{T_8} \frac{\left(\frac{T_3}{T_8} - \frac{T_{11}}{T_1}\right)}{\left(\frac{T_9}{T_1} - \frac{T_4}{T_8}\right)} \right] \frac{n^2}{Et'r^2}$$

$$X^*_{12} = \left[-\frac{T_7}{T_8} - \frac{T_4}{T_8} \; Et' \, X^*_{22} \right] \frac{n}{Et'r}$$

$$X^*_{21} = \frac{\left(\frac{T_3}{T_8} - \frac{T_{11}}{T_1}\right)}{\left(\frac{T_9}{T_1} - \frac{T_4}{T_8}\right)} \cdot \frac{n}{Et'r} \quad ; \quad X^*_{22} = \frac{\left(\frac{T_7}{T_8} - \frac{T_3}{T_1}\right)}{\left(\frac{T_9}{T_1} - \frac{T_4}{T_8}\right)} \cdot \frac{1}{Et'}$$

The identities of equations (26) have been freely used in deriving these four coefficients. It can be shown that $X^*_{12} = - X^*_{21}$ and that the coefficents converge to the corresponding Z_{ik} as ℓ / L_c increases. These facts are to be expected from the previous work.

Except for the difference in the coefficients of the equations governing the finite length of shell, the equations of this system are the same as for the case of a free end at a finite distance from the loaded frame.

Hence, using equations (127) in place of (122) and carrying through the operations given for the case of the free end, we arrive at the Input Impedance:

$$\frac{\bar{v}_n(o)}{\Delta q_n(o)} = \left\{ X^*_{11} \quad X^*_{12} \right\} \begin{bmatrix} X^*_{11} + Z_{11} & X^*_{12} - Z_{12} \\ X^*_{21} - Z_{21} & X^*_{22} + Z_{22} \end{bmatrix}^{-1} \begin{vmatrix} Z_{11} \\ -Z_{21} \end{vmatrix} \qquad (128)$$

i.e., the X^*_{ik} replace the X_{ik} of the case of a free end at $x = \ell$.

The Input Impedance, substituted in equation (4), gives f(n). Figure 11 shows f(2) plotted against ℓ / L_c .

It is seen in figure 11 that this is the only case where the effects of a discontinuity extend beyond $\ell / L_c = 1.00$.

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Plane of Symmetry at a Finite Distance From the Loaded Frame

As mentioned in the case of the built-in end, the only differences between this case and that of the free end at $x = \ell$ are the boundary conditions at that point.

Imposing the conditions $\bar{q}_n(\ell) = \bar{u}_n(\ell) = 0$ on equations (19) enables the following stress-displacement relations to be found for the finite length of frame at $x = 0$.

$$\begin{vmatrix} \bar{v}_n(o) \\ \bar{u}_n(o) \end{vmatrix} = \begin{bmatrix} X'_{11} & X'_{12} \\ X'_{21} & X'_{22} \end{bmatrix} \begin{vmatrix} \bar{q}_n(o) \\ \bar{p}_n(o) \end{vmatrix} \qquad (129)$$

where

$$X'_{11} = \left[-\frac{T_1}{T_6} - \frac{T_2}{T_6} \frac{\left(\frac{T_3}{T_8} - \frac{T_1}{T_6}\right)}{\left(\frac{T_2}{T_6} - \frac{T_4}{T_8}\right)} \right] \frac{n^2}{Et'r^2} \quad ; \quad X_{21} = \frac{\left(\frac{T_3}{T_8} - \frac{T_1}{T_6}\right)}{\left(\frac{T_2}{T_6} - \frac{T_4}{T_8}\right)} \cdot \frac{n}{Et'r}$$

$$X'_{12} = \left[-\frac{T_5}{T_6} - \frac{T_2}{T_6} \frac{\left(\frac{T_7}{T_8} - \frac{T_5}{T_6}\right)}{\left(\frac{T_2}{T_6} - \frac{T_4}{T_8}\right)} \right] \frac{n}{Et'r} \quad ; \quad X'_{22} = \frac{\left(\frac{T_7}{T_8} - \frac{T_5}{T_6}\right)}{\left(\frac{T_2}{T_6} - \frac{T_4}{T_8}\right)} \cdot \frac{1}{Et'}$$

It can easily be shown that $X'_{12} = -X'_{21}$ and that X'_{ik} converges to the corresponding Z_{ik} as ℓ/L_c becomes large:

Replacing the coefficients X_{ik} of equation (125) by the X_{ik} of equation (130), we have the equation for the Input Impedance.

$$\frac{\bar{v}_n(o)}{\Delta\bar{q}_n(o)} = \left\{ X'_{11} \quad X'_{12} \right\} \begin{bmatrix} (X'_{11} + Z_{11}) & (X'_{12} - Z_{12}) \\ (X'_{21} - Z_{21}) & (X'_{22} + Z_{22}) \end{bmatrix}^{-1} \begin{vmatrix} Z_{11} \\ -Z_{21} \end{vmatrix} \qquad (130)$$

The $f(n)$ required is obtained by substituting equation (130) into equation (4). The values of $f(2)$ are given in figure 11 for $L_r/L_c = 0.4$.

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Plane of Antisymmetry at a Finite Distance From the Loaded Frame

Apart from the different boundary conditions at $x = \ell$, this case is the same as that of the free end at a finite distance from the loaded frame. If the boundary conditions $\bar{p}_n(\ell) = \bar{v}_n(\ell) = 0$ are applied to equations (19) for the finite length of shell, we obtain the following stress-displacement relationships at the loaded frame:

$$\begin{vmatrix} \bar{v}_n(o) \\ \bar{u}_n(o) \end{vmatrix} = \begin{bmatrix} X''_{11} & X''_{12} \\ X''_{21} & X''_{22} \end{bmatrix} \begin{vmatrix} \bar{q}_n(o) \\ \bar{p}_n(o) \end{vmatrix} \tag{131}$$

where

$$X''_{11} = \left[-\frac{T_9}{T_2} - \frac{T_8}{T_2} \frac{\left(\frac{T_{11}}{T_1} - \frac{T_9}{T_2}\right)}{\left(\frac{T_8}{T_2} - \frac{T_9}{T_1}\right)} \right] \frac{n^2}{Et'r^2} \quad ; \quad X''_{21} = \frac{\left(\frac{T_{11}}{T_1} - \frac{T_9}{T_2}\right)}{\left(\frac{T_8}{T_2} - \frac{T_9}{T_1}\right)} \cdot \frac{n}{Et'r}$$

$$X''_{12} = - X''_{21} \quad ; \quad X''_{22} = \frac{\left(\frac{T_3}{T_1} - \frac{T_4}{T_2}\right)}{\left(\frac{T_8}{T_2} - \frac{T_9}{T_1}\right)} \cdot \frac{1}{Et'}$$

Note that the identities of equation (26) have been used in the presentation of these coefficients.

Hence, the Input Impedance can be written by replacing the X_{ik}'s of equation (125) by the corresponding X_{ik}'s of equation (131).

$$\frac{\bar{v}_n(o)}{\Delta\bar{q}_n(o)} = \begin{Bmatrix} X''_{11} & X''_{12} \end{Bmatrix} \begin{bmatrix} (X''_{11} + Z_{11}) & (X''_{12} - Z_{12}) \\ (X''_{21} - Z_{21}) & (X''_{22} + Z_{22}) \end{bmatrix}^{-1} \begin{vmatrix} Z_{11} \\ -Z_{21} \end{vmatrix} \tag{132}$$

Substitute equation (132) into equation (4) to give $f(n)$ for this problem. Figure 11 shows $f(2)$ as a function of ℓ/L_c for $L_r/L_c = 0.4$.

Additional Relationships Between Coefficients of Transmission Matrix of a Finite Length of Shell

The six identities of equations (26) were derived as a result of reciprocity argument, and it is mentioned there that at least three additional relationships exist between the coefficients.

It can be shown that in the square matrices of equations (122), (127), (129), and (131), $X_{12} = -X_{21}$; $X^*_{12} = -X^*_{21}$; $X'_{12} = -X'_{21}$ and $X''_{12} = -X''_{21}$. Using the values of the coefficients given in the four equations mentioned, the following relationships can be shown to exist between the transmission coefficients.

$$
\left.
\begin{aligned}
T_4 T_5 - T_2 T_7 + T_3 T_6 - T_1 T_8 &= 0 \\
T_4 T_9 - T_3 T_8 + T_2 T_{11} - T_1 T_9 &= 0 \\
T_5^2 - T_2 T_4 + T_6 T_9 - T_1 T_2 &= 0 \\
T_3 T_4 - T_7 T_9 + T_1 T_3 - T_8 T_{11} &= 0
\end{aligned}
\right\} \qquad (13
$$

These equations can be usefully employed as a check on numerical work.

Lockheed Aircraft Corporation,
California Division,
Burbank, Calif., October 1959.

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3. MacNeal, R. H.: "Vibrations of Composite Systems," ARDC Report no. 4 under contract AF 18(600)-669, California Institute of Technology 1954.

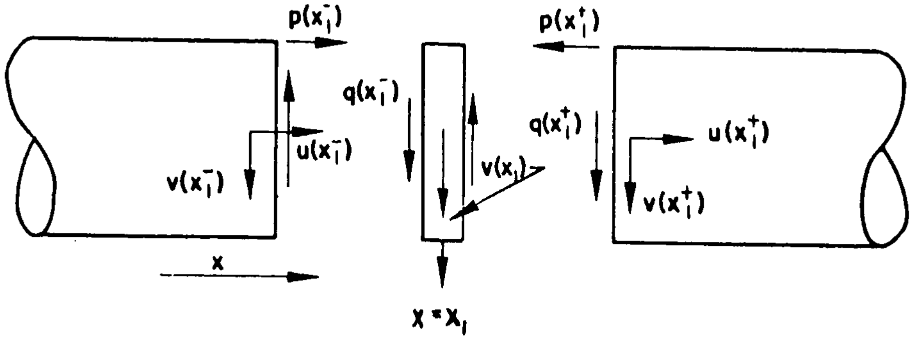

Figure 1. - General discontinuity in the shell.

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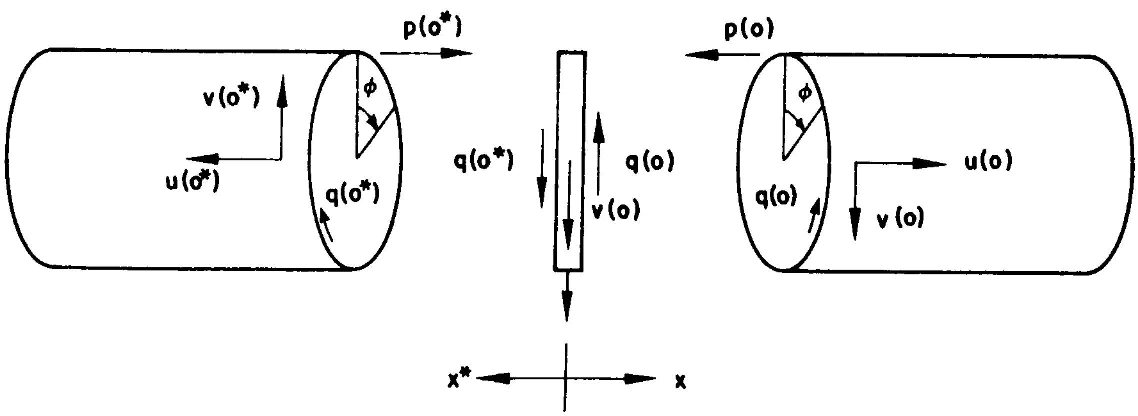

Figure 2. - Exploded view of shell-frame intersection to illustrate the boundary value problem.

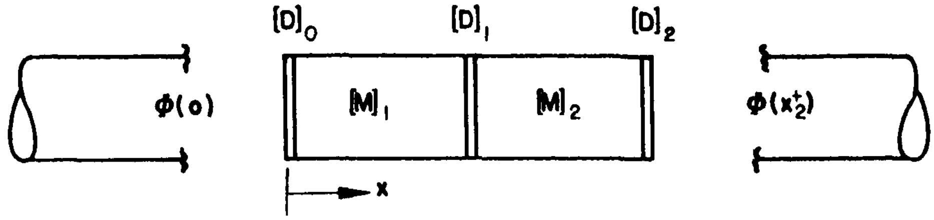

Figure 3. - A shell with many stiff frames.

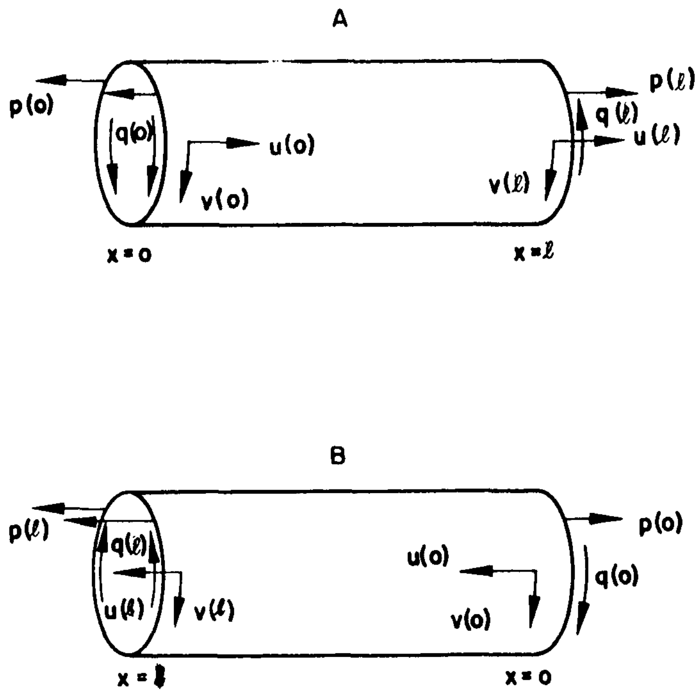

Figure 4. - Loads and deflections in the shell used in the derivation of the elements of the transmission matrix.

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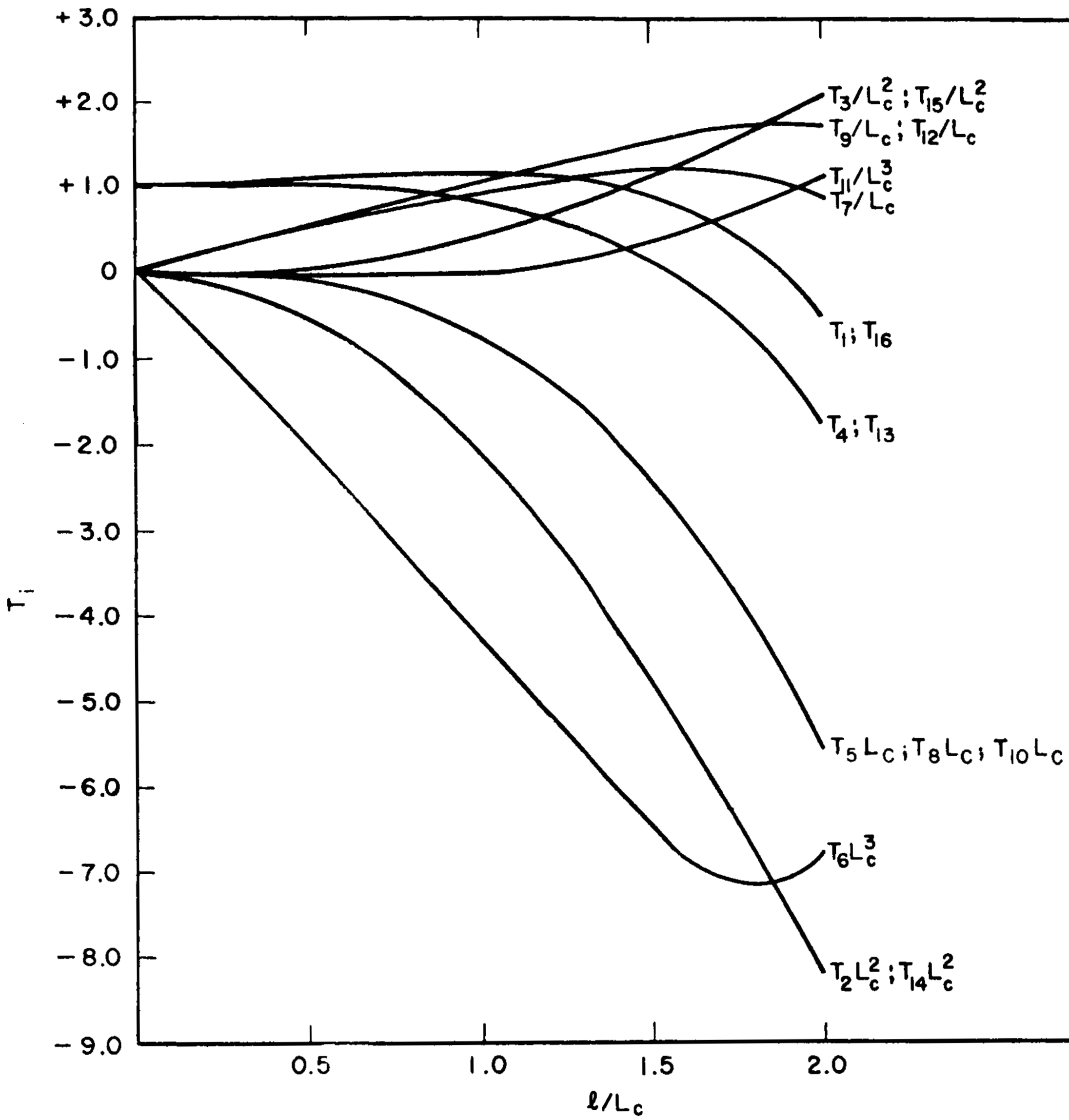

Figure 5. - Transmission constants for a finite length of shell $n = 2$ and $L_r/L_c = 0.4$.

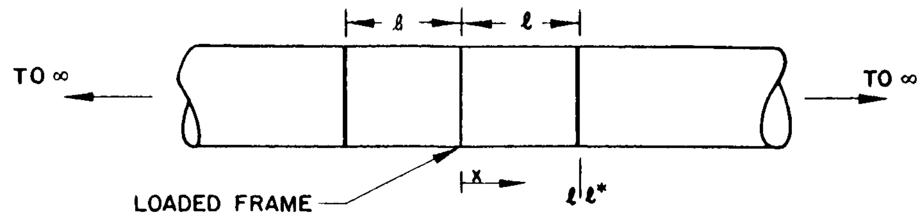

Figure 6. - Two rigid bulkheads symmetrically placed about the loaded frame.

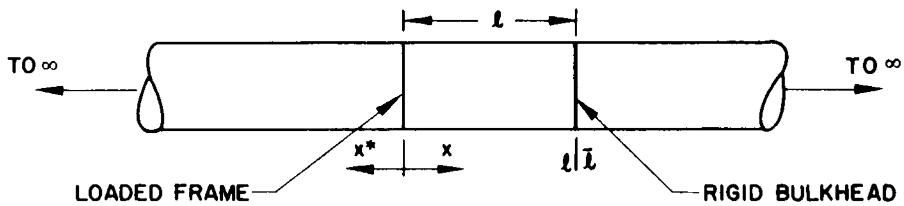

Figure 7. - A rigid bulkhead on one side of the loaded frame.

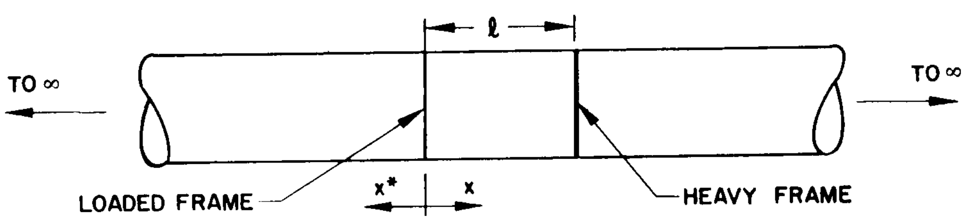

Figure 8. - A heavy frame on one side of the loaded frame.

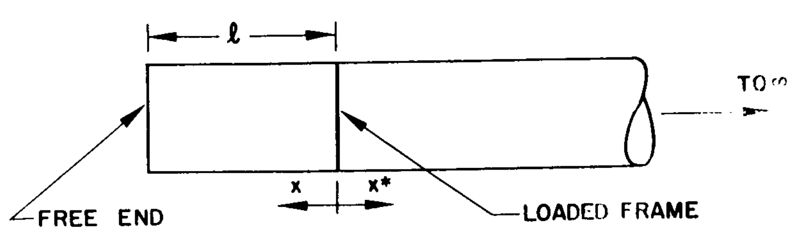

Figure 9. - A free end at a finite distance from the loaded frame.

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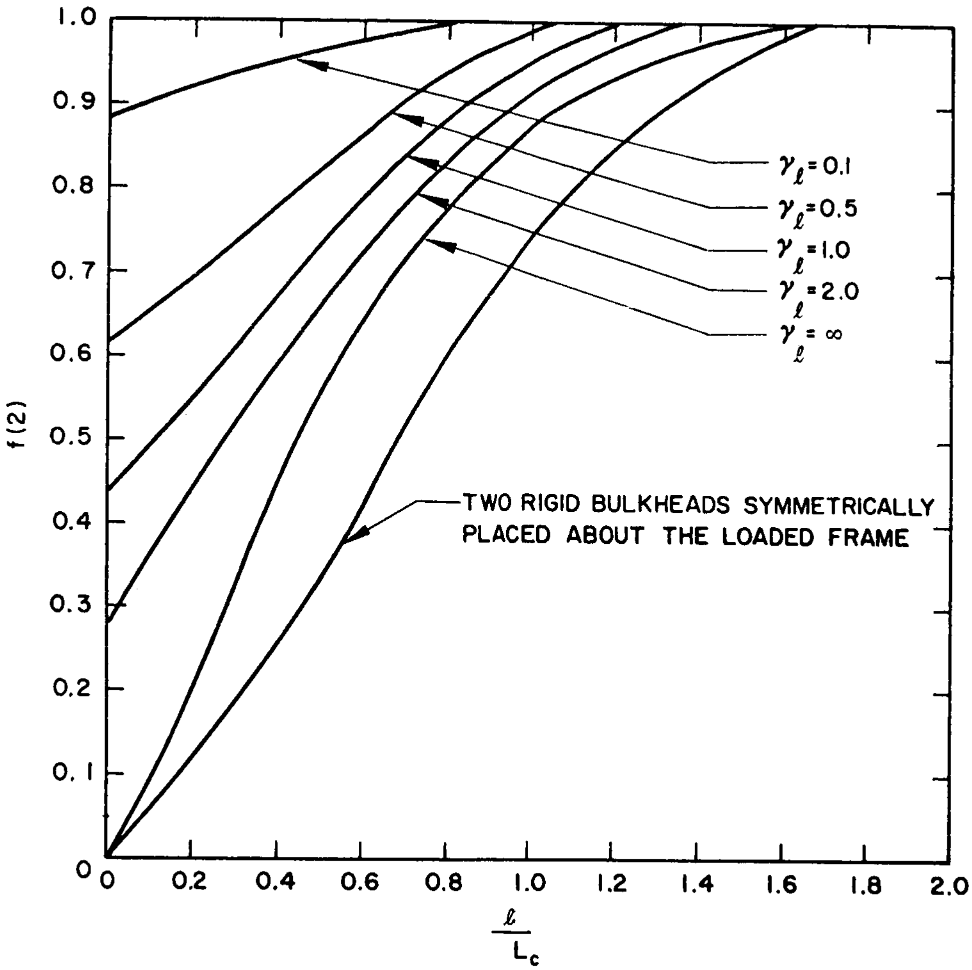

Figure 10. - A single frame on one side of loaded frame or two rigid bulkheads symmetrically placed about the loaded frame curves of f(2) and f(3). $L_r/L_c = 0.4$.

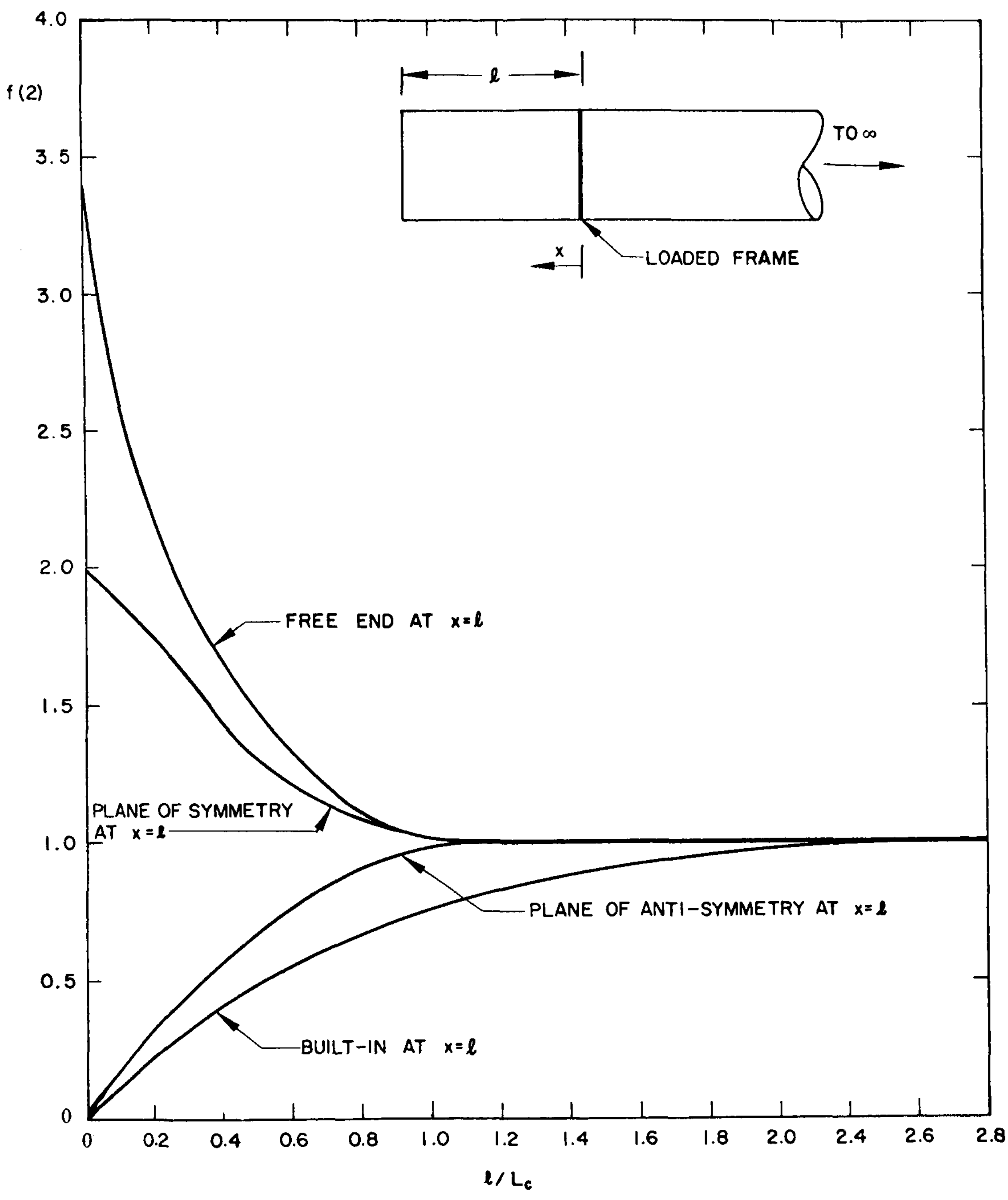

Figure 11.- Finite length of shell on one side of loaded frame f(2) v ℓ/L_c for various boundary conditions at $x = \ell$. $L_r/L_c = 0.4$.